UNIT 1:

INTRODUCTION TO SIMULATION

Simulation

A Simulation is the imitation of the operation of a real-world process or system over time

Brief Explanation

- The behavior of a system as it evolves over time is studied by developing a simulation model.
- This model takes the form of a set of assumptions concerning the operation of the system.

The assumptions are expressed in

- Mathematical relationships
- Logical relationships
- Symbolic relationships

Between the entities of the system.

Measures of performance

The model solved by mathematical methods such as differential calculus, probability theory, algebraic methods have the solution usually consists of one or more numerical parameters which are called measures of performance.

1.1 When Simulation is the Appropriate Tool

- Simulation enables the study of and experimentation with the internal interactions of a complex system, or of a subsystem within a complex system.
- Informational, organizational and environmental changes can be simulated and the effect of those alternations on the model‘s behavior can be observer.
- The knowledge gained in designing a simulation model can be of great value toward suggesting improvement in the system under investigation.
- By changing simulation inputs and observing the resulting outputs, valuable insight may be obtained into which variables are most important and how variables interact.
- Simulation can be used as a pedagogical device to reinforce analytic solution methodologies.
- Simulation can be used to experiment with new designs or policies prior to implementation, so as to prepare for what may happen.
- Simulation can be used to verify analytic solutions.
- By simulating different capabilities for a machine, requirements can be determined.
- Simulation models designed for training, allow learning without the cost and disruption of on-the-job learning.
- Animation shows a system in simulated operation so that the plan can be visualized.
- The modern system(factory, water fabrication plant, service organization, etc) is so complex that the interactions can be treated only through simulation.

When Simulation is Not Appropriate

- Simulation should be used when the problem cannot be solved using common sense.
- Simulation should not be used if the problem can be solved analytically.
- Simulation should not be used, if it is easier to perform direct experiments.
- Simulation should not be used, if the costs exceeds savings.
- Simulation should not be performed, if the resources or time are not available.
- If no data is available, not even estimate simulation is not advised.
- If there is not enough time or the person are not available, simulation is not appropriate.
- If managers have unreasonable expectation say, too much soon – or the power of simulation is over estimated, simulation may not be appropriate.
- If system behavior is too complex or cannot be defined, simulation is not appropriate.

1.2**Advantages of Simulation**

- Simulation can also be used to study systems in the design stage.
- Simulation models are run rather than solver.
- New policies, operating procedures, decision rules, information flow, etc can be explored without disrupting the ongoing operations of the real system.
- New hardware designs, physical layouts, transportation systems can be tested without committing resources for their acquisition.
- Hypotheses about how or why certain phenomena occur can be tested for feasibility.
- Time can be compressed or expanded allowing for a speedup or slowdown of the phenomena under investigation.
- Insight can be obtained about the interaction of variables.
- Insight can be obtained about the importance of variables to the performance of the system.
- Bottleneck analysis can be performed indication where work-inprocess, information materials and so on are being excessively delayed.\
- A simulation study can help in understanding how the system operates rather than how individuals think the system operates.
- ―what-if‖ questions can be answered. Useful in the design of new systems.

Disadvantages of simulation

- Model building requires special training.
- Simulation results may be difficult to interpret.
- Simulation modeling and analysis can be time consuming and expensive.
- Simulation is used in some cases when an analytical solution is possible or even preferable.

Applications of Simulation

Manufacturing Applications

1. Analysis of electronics assembly operations
2. Design and evaluation of a selective assembly station for highprecision scroll compressor shells.
3. Comparison of dispatching rules for semiconductor manufacturing using large facility models.
4. Evaluation of cluster tool throughput for thin-film head production.
5. Determining optimal lot size for a semiconductor backend factory.
6. Optimization of cycle time and utilization in semiconductor test manufacturing.
7. Analysis of storage and retrieval strategies in a warehouse.
8. Investigation of dynamics in a service oriented supply chain.
9. Model for an Army chemical munitions disposal facility.

Semiconductor Manufacturing

1. Comparison of dispatching rules using large-facility models.
2. The corrupting influence of variability.
3. A new lot-release rule for wafer fabs.
4. Assessment of potential gains in productivity due to proactive retied management.
5. Comparison of a 200 mm and 300 mm X-ray lithography cell.
6. Capacity planning with time constraints between operations.
7. 300 mm logistic system risk reduction.

Construction Engineering

1. Construction of a dam embankment.
2. Trench less renewal of underground urban infrastructures.
3. Activity scheduling in a dynamic, multiproject setting.
4. Investigation of the structural steel erection process.
5. Special purpose template for utility tunnel construction.

Military Applications

1. Modeling leadership effects and recruit type in a Army recruiting station.
2. Design and test of an intelligent controller for autonomous underwater vehicles.
3. Modeling military requirements for nonwarfighting operations.
4. Multitrajectory performance for varying scenario sizes.
5. Using adaptive agents in U.S. Air Force retention.

Logistics. Transportation and Distribution Applications

1. Evaluating the potential benefits of a rail-traffic planning algorithm.
2. Evaluating strategies to improve railroad performance.
3. Parametric Modeling in rail-capacity planning.
4. Analysis of passenger flows in an airport terminal.
5. Proactive flight-schedule evaluation.
6. Logistic issues in autonomous food production systems for extended duration space exploration.
7. Sizing industrial rail-car fleets.
8. Production distribution in newspaper industry.
9. Design of a toll plaza
10. Choosing between rental-car locations.
11. Quick response replenishment.

Business Process Simulation

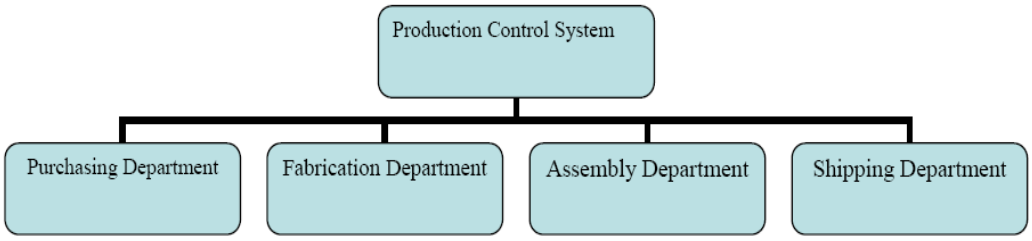
1. Impact of connection bank redesign on airport gate assignment.
2. Product development program planning.
3. Reconciliation of business and system modeling.
4. Personal forecasting and strategic workforce planning.

Human Systems

1. Modeling human performance in complex systems.
2. Studying the human element in out traffic control.

1.3 Systems

A system is defined as an aggregation or assemblage of objects joined in some regular interaction or interdependence toward the accomplishment of some purpose.



Example : Production System

In the above system there are certain distinct objects, each of which possesses properties of interest. There are also certain interactions occurring in the system that cause changes in the system.

Components of a System

Entity : An entity is an object of interest in a system.

Ex: In the factory system, departments, orders, parts and products are The entities.

Attribute

An attribute denotes the property of an entity.

Ex: Quantities for each order, type of part, or number of machines in a Department are attributes of factory system.

Activity

Any process causing changes in a system is called as an activity.

Ex: Manufacturing process of the department.

State of the System

The state of a system is defined as the collection of variables necessary to describe a system at any time, relative to the objective of study. In other words, state of the system mean a description of all the entities, attributes and activities as they exist at one point in time.

Event

An event is defined as an instaneous occurrence that may change the state of the system.

System Environment

The external components which interact with the system and produce necessary changes are said to constitute the system environment. In modeling systems, it is necessary to decide on the boundary between the system and its environment. This decision may depend on the purpose of the study.

Ex: In a factory system, the factors controlling arrival of orders may be considered to be outside the factory but yet a part of the system environment. When, we consider the demand and supply of goods, there is certainly a relationship between the factory output and arrival of orders. This relationship is considered as an activity of the system.

Endogenous System

The term endogenous is used to describe activities and events occurring within a system. Ex: Drawing cash in a bank.

Exogenous System

The term exogenous is used to describe activities and events in the environment that affect the system. Ex: Arrival of customers.

Closed System

A system for which there is no exogenous activity and event is said to be a closed. Ex: Water in an insulated flask.

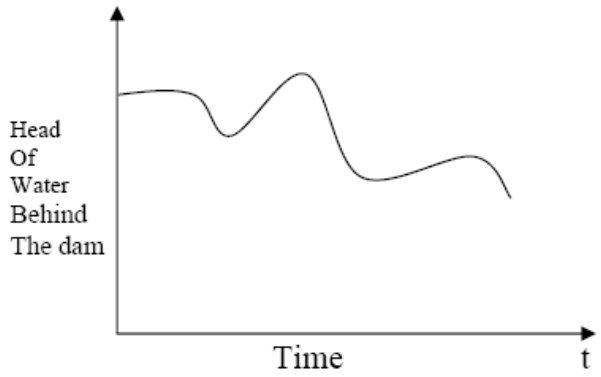
Open system

A system for which there is exogenous activity and event is said to be a open. Ex: Bank system.

1.4**Discrete and Continuous Systems**

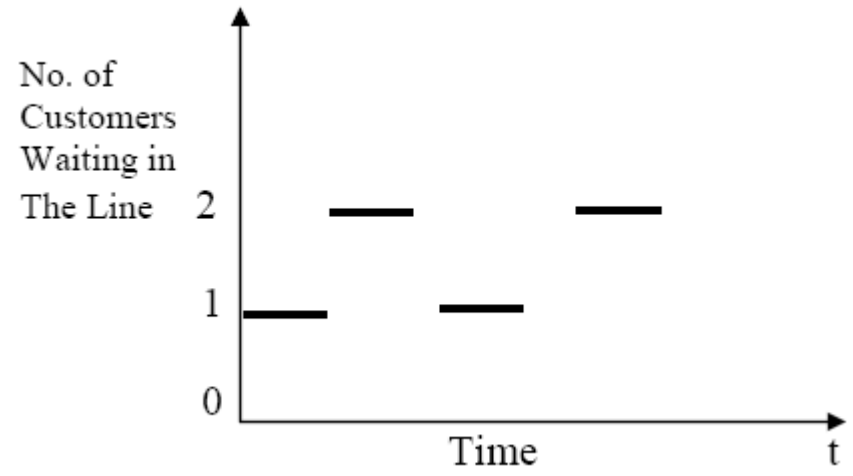
Continuous Systems

Systems in which the changes are predominantly smooth are called continuous system. Ex: Head of a water behind a dam.



Discrete Systems

Systems in which the changes are predominantly discontinuous are called discrete systems. Ex: Bank – the number of customer‘s changes only when a customer arrives or when the service provided a customer is completed.



1.5**Model of a system**

A model is defined as a representation of a system for the purpose of studying the system. It is necessary to consider only those aspects of the system that affect the problem under investigation. These aspects are represented in a model, and by definition it is a simplification of the system.

Types of Models

The various types models are

- Mathematical or Physical Model
- Static Model
- Dynamic Model
- Deterministic Model
- Stochastic Model
- Discrete Model
- Continuous Model

Mathematical Model

Uses symbolic notation and the mathematical equations to represent a system.

Static Model

Represents a system at a particular point of time and also known as Monte-Carlo simulation.

Dynamic Model

Represents systems as they change over time. Ex: Simulation of a bank

Deterministic Model

Contains no random variables. They have a known set of inputs which will result in a unique set of outputs. Ex: Arrival of patients to the Dentist at the scheduled appointment time.

Stochastic Model

Has one or more random variable as inputs. Random inputs leads to random outputs. Ex: Simulation of a bank involves random interarrival and service times.

Discrete and Continuous Model

Used in an analogous manner. Simulation models may be mixed both with discrete and continuous. The choice is based on the characteristics of the system and the objective of the study.

Discrete-Event System Simulation

Modeling of systems in which the state variable changes only at a discrete set of points in time. The simulation models are analyzed by numerical rather than by analytical methods. Analytical methods employ the deductive reasoning of mathematics to solve the model. Eg: Differential calculus can be used to determine the minimum cost policy for some inventory models.

Numerical methods use computational procedures and are _runs‘, which is generated

based on the model assumptions and observations are collected to be analyzed and to estimate the true system performance measures. Real-world simulation is so vast, whose runs are conducted with the help of computer. Much insight can be obtained by simulation manually which is applicable for small systems.

1.6 **Steps in a Simulation study**

1. Problem formulation

Every study begins with a statement of the problem, provided by policy makers. Analyst ensures its clearly understood. If it is developed by analyst policy makers should understand and agree with it.

2.**Setting of objectives and overall project plan**

The objectives indicate the questions to be answered by simulation. At this point a determination should be made concerning whether simulation is the appropriate methodology. Assuming it is appropriate, the overall project plan should include

- A statement of the alternative systems
- A method for evaluating the effectiveness of these alternatives
- Plans for the study in terms of the number of people involved
- Cost of the study
- The number of days required to accomplish each phase of the work with the anticipated results.

3.**Model conceptualization**

The construction of a model of a system is probably as much art as science. The art of modeling is enhanced by an ability

- To abstract the essential features of a problem
- To select and modify basic assumptions that characterize the system
- To enrich and elaborate the model until a useful approximation results

Thus, it is best to start with a simple model and build toward greater complexity. Model conceptualization enhance the quality of the resulting model and increase the confidence of the model user in the application of the model.

4.**Data collection**

There is a constant interplay between the construction of model and the collection of needed input data. Done in the early stages.

Objective kind of data are to be collected.

5.**Model translation**

Real-world systems result in models that require a great deal of information storage and computation. It can be programmed by using simulation languages or special purpose simulation software.

Simulation languages are powerful and flexible. Simulation software models development time can be reduced.

6.**Verified**

It pertains to he computer program and checking the performance. If the input parameters and logical structure and correctly represented, verification is completed.

7.**Validated**

It is the determination that a model is an accurate representation of the real system. Achieved through calibration of the model, an iterative process of comparing the model to actual system behavior and the discrepancies between the two.

8.**Experimental Design**

The alternatives that are to be simulated must be determined. Which alternatives to simulate may be a function of runs. For each system design, decisions need to be made concerning

- Length of the initialization period
- Length of simulation runs
- Number of replication to be made of each run

9.**Production runs and analysis**

They are used to estimate measures of performance for the system designs that are being simulated.

10.**More runs**

Based on the analysis of runs that have been completed. The analyst determines if additional runs are needed and what design those additional experiments should follow.

11.**Documentation and reporting**

Two types of documentation.

- Program documentation
- Process documentation

Program documentation

Can be used again by the same or different analysts to understand how the program operates. Further modification will be easier. Model users can change the input parameters for better performance.

Process documentation

Gives the history of a simulation project. The result of all analysis should be reported clearly and concisely in a final report. This enables to review the final formulation and alternatives, results of the experiments and the recommended solution to the problem. The final report provides a vehicle of certification.

12. Implementation

Success depends on the previous steps. If the model user has been thoroughly involved and understands the nature of the model and its outputs, likelihood of a vigorous implementation is enhanced.

The simulation model building can be broken into 4 phases.

I Phase

- Consists of steps 1 and 2
- It is period of discovery/orientation
- The analyst may have to restart the process if it is not fine-tuned
- Recalibrations and clarifications may occur in this phase or another phase.

II Phase

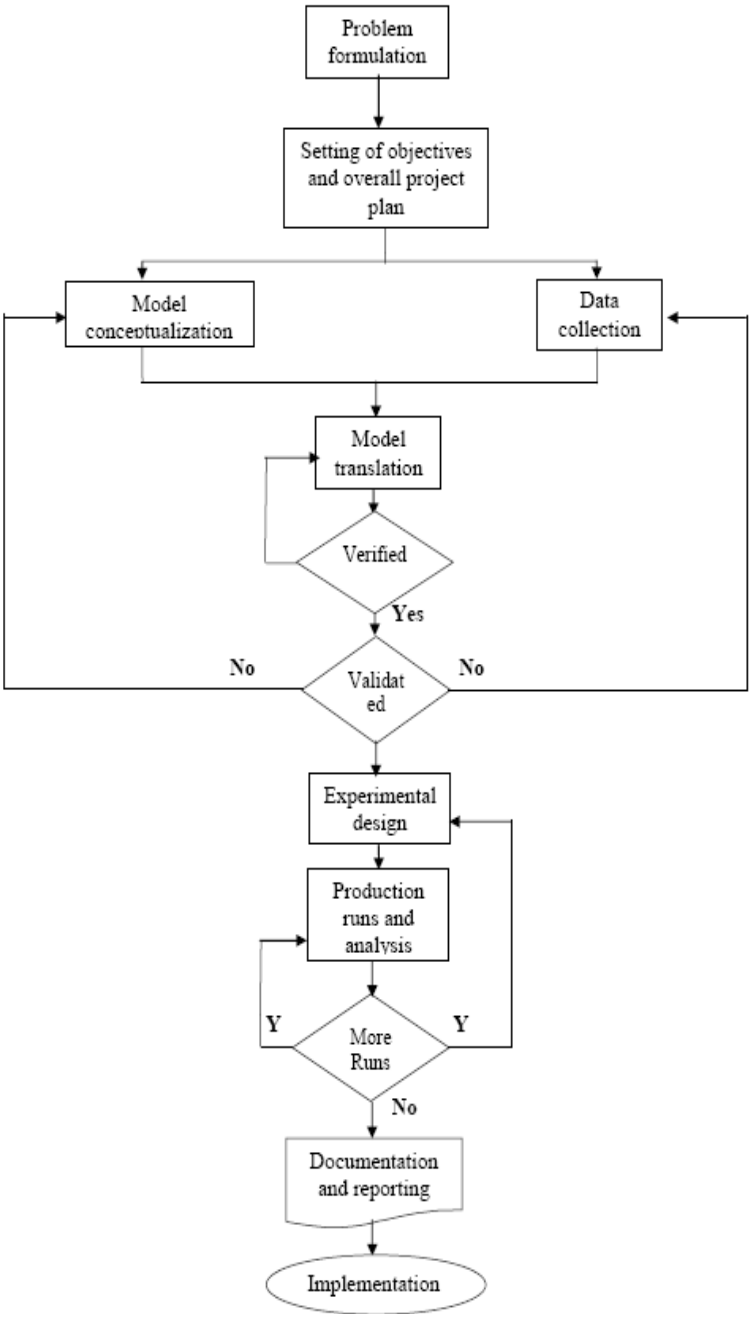
- Consists of steps 3,4,5,6 and 7
- A continuing interplay is required among the steps
- Exclusion of model user results in implications during implementation

III Phase

- Consists of steps 8,9 and 10
- Conceives a thorough plan for experimenting
- Discrete-event stochastic is a statistical experiment
- The output variables are estimates that contain random error and therefore proper statistical analysis is required.

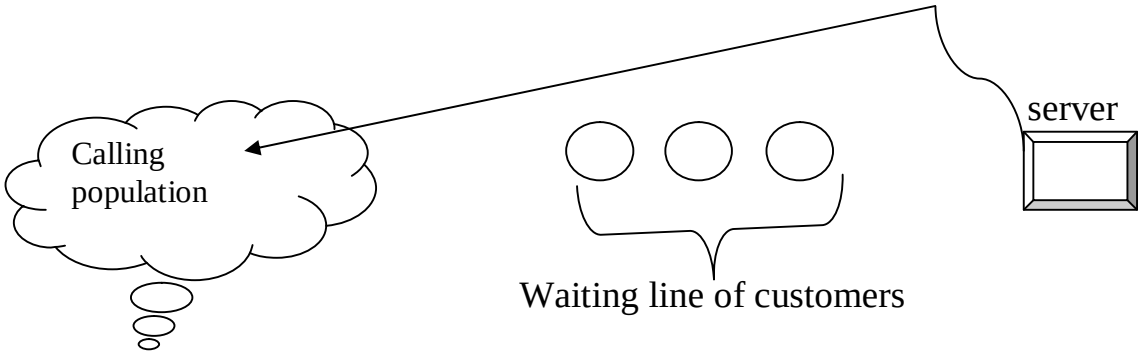
IV Phase

- Consists of steps 11 and 12
- Successful implementation depends on the involvement of user and every steps successful completion.



1.7**SIMULATION EXAMPLES**

- Simulation is often used in the analysis of queueing models. In a simple typical queueing model, shown in fig 1, customers arrive from time to time and join a queue or waiting line, are eventually served, and finally leave the system.
- The term "customer" refers to any type of entity that can be viewed as requesting "service" from a system.



Simple Queue Model

Characteristics of Queueing Systems

- The key elements, of a queueing system are the customers and servers. The term "customer" can refer to people, machines, trucks, mechanics, patients—anything that arrives at a facility and requires service.
- The term "server" might refer to receptionists, repairpersons, CPUs in a computer, or washing machines….any resource (person, machine, etc. which provides the requested service.
- Table 1 lists a number of different queuing systems.

System	*Customers*	*Server(s)*
Reception desk	People	Receptionist
Repair facility	Machines	Repairperson
Garage	Trucks	Mechanic
Tool crib	Mechanics	Tool-crib clerk
Hospital	Patients	Nurses
Warehouse	Pallets	Crane
Airport	Airplanes	Runway
Production line	Cases	Case packer
Warehouse	Orders	Order picker
Road network	Cars	Traffic light
Grocery	Shoppers	Checkout station
Laundry	Dirty linen	Washing machines/dryers
Job shop	Jobs	Machines/workers
Lumberyard	Trucks	Overhead crane
Saw mill	Logs	Saws
Computer	Jobs	CPU, disk, tapes
Telephone	Calls	Exchange
Ticket office	Football fans	Clerk
Mass transit	Riders	Buses, trains

Table 1: Examples of Queueing Systems

➢ The elements of a queuing system are:-

The Calling Population:-

✓ The population of potential customers, referred to as the calling population, may beassumed to be finite or infinite.

✓ For example, consider a bank of 5 machines that are curing tires. After an interval of time, a machine automatically opens and must be attended by a worker who removes the tire and puts an uncured tire into the machine. The machines are the "customers", who "arrive" at the instant they automatically open. The worker is the "server", who "serves" an open machine as soon as possible. The calling population is finite, and consists of the five machines.

- ✓ In systems with a large population of potential customers, the calling population is usually assumed to be finite or infinite. Examples of infinite populations include the potential customers of a restaurant, bank, etc.

- ✓ The main difference between finite and infinite population models is how the arrival rate is defined. In an infinite-population model, the arrival rate is not affected by the number of customers who have left the calling population and joined the queueing system. On the other hand, for finite calling population models, the arrival rate to the queueing system does depend on the number of customers being served and waiting.

System Capacity:-

- ✓ In many queueing systems there is a limit to the number of customers that may be in the waiting line or system. For example, an automatic car wash may have room for only 10 cars to wait in line to enter the mechanism.

- ✓ An arriving customer who finds the system full does not enter but returns immediately to the calling population.

- ✓ Some systems, such as concert ticket sales for students, may be considered as having unlimited capacity. There are no limits on the number of students allowed to wait to purchase tickets.

- ✓ When a system has limited capacity, a distinction is made between the arrival rate (i.e., the number of arrivals per time unit) and the effective arrival rate (i.e., the number who arrive and enter the system per time unit).

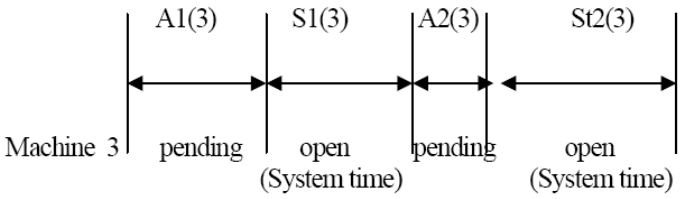
The Arrival Process:-

- ✓ Arrival process for infinite-population models is usually characterized in terms of interarrival times of successive customers. Arrivals may occur at scheduled times or at random times. When at random times, the interarrival times are usually characterized by a probability distribution

- ✓ The most important model for random arrivals is the Poisson arrival process. If An represents the interarrival time between customer n-1 and customer n (A1 is the actual arrival time of the first customer), then for a Poisson arrival process. An is exponentially distributed with mean I/λ time Units. The arrival rate is λ customers per time unit. The number of arrivals in a time interval of length t, say N(t) , has the Poisson distribution with mean λt customers.
- ✓ The Poisson arrival process has been successfully employed as a model of the arrival of people to restaurants, drive-in banks, and other service facilities.

- ✓ A second important class of arrivals is the scheduled arrivals, such as patients to a physician's office or scheduled airline flight arrivals to an airport. In this case, the interarrival times [An , n = 1,2,. . . } may be constant, or constant plus or minus a small random amount to represent early or late arrivals.
- ✓ A third situation occurs when at least one customer is assumed to always be present in the queue, so that the server is never idle because of a lack of customers. For example, the "customers" may represent raw material for a product, and sufficient raw material is assumed to be always available.

- ✓ For finite-population models, the arrival process is characterized in a completely different fashion. Define a customer as pending when that customer is outside the queueing system and a member of the potential calling population.
- ✓ Runtime of a given customer is defined as the length of time from departure from the queueing system until that customer‘s next arrival to the queue.
- ✓ Let $A_i^{(i)}, A_2^{(i)}$,...be the successive runtimes of customer /, and let $S_{(i)}^{1}, S_{(i)}^{2}$...be the corresponding successive system times; that is, $S^{(i)}_n$ is the total time spent in the system



First arrival of machine 3 Second arrival of machine 3

by customer i during the nth visit. Figure 2 illustrates these concepts for machine 3 in the tire-curing example. The total arrival process is the superposition of the arrival times of all customers.

Fig 2 shows the first and second arrival of machine 3.

Fig 2: Arrival process for a finite-population model.

- ✓ One important application of finite population models is the machine repair problem. The machines are the customers and a runtime is also called time to failure. When a machine fails, it "arrives" at the queuing system (the repair facility) and remains there until it is "served" (repaired). Times to failure for a given class of machine have been characterized by the exponential, the Weibull, and the gamma distributions. Models with an exponential runtime are sometimes analytically tractable.

Queue Behavior and Queue Discipline:-

- ✓ Queue behavior refers to customer actions while in a queue waiting for service to begin. In some situations, there is a possibility that incoming customers may balk (leave when they see that the line is too long), renege (leave after being in the line when they see that the line is moving too slowly), or jockey (move from one line to another if they think they have chosen a slow line).

- ✓ Queue discipline refers to the logical ordering of customers in a queue and determineswhich customer will be chosen for service when a server becomes free.
  - Common queue disciplines include first-in, first-out (FIFO); last-in firstout (LIFO); service in random order (SIRO); shortest processing time first |(SPT) and service according to priority (PR).

  - In a job shop, queue disciplines are sometimes based on due dates and on expected processing time for a given i type of job. Notice that a FIFO queue discipline implies that services begin in the same order as arrivals, but that customers may leave the system in a different order because of differentlength service times.

- ***Service Times and the Service Mechanism:-***

- ✓ The service times of successive arrivals are denoted by S1, S2, S3…They may be constant or of random duration. The exponential,Weibull, gamma, lognormal, and truncated normal distributions have all been used successfully as models of service times in different situations.

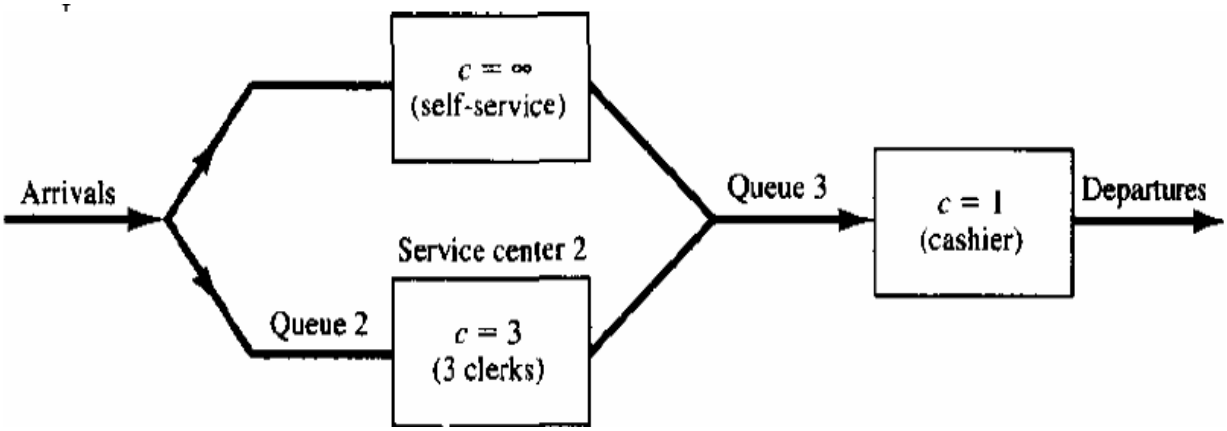
- ✓ Sometimes services may be identically distributed for all customers of a given type or class or priority, while customers of different types may have completely different service-time distributions. In addition, in some systems, service times depend upon the time of day or the length of the waiting line. For example, servers may work faster than

usual when the waiting line is long, thus effectively reducing the service times.

- ✓ A queueing system consists of a number of service centers and interconnecting queues. Each service center consists of some number of servers, c, working in parallel; that is, upon getting to the head of the line, a customer takes the first available server. Parallel service mechanisms are either single server (c = 1), multiple server (1 < c < ∞), or unlimited servers (c= ∞). (A self-service facility is usually characterized as having an unlimited number of servers.)

Example 1:-

Consider a discount warehouse where customers may either serve themselves; or wait



of three clerks, and finally leave after paying a single cashier. The system

is represented by the flow diagram in figure 1 below:

Figure 1: Discount warehouse with three service centers

The subsystem, consisting of queue 2 and service center 2, is shown in more detail in figure 2 below. Other variations of service mechanisms include batch service (a server serving several customers simultaneously) or a customer requiring several servers simultaneously.

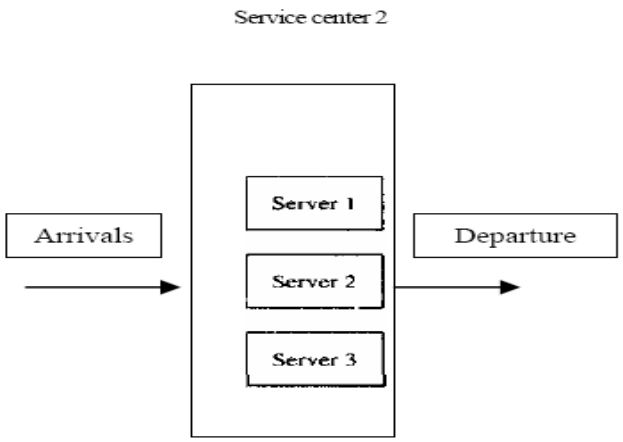


Figure 2: Service center 2, with c = 3 parallel servers.

- Example 2:-A candy manufacturer has a production line which consists of three machines separated by inventory-in-process buffers. The first machine makes and wraps the individual pieces of candy, the second packs 50 pieces in a box, and the third seals and wraps the box. The two inventory buffers have capacities of 1000 boxes each. As illustrated by Figure 3, the system is modeled as having three service centers, each center having c = 1 server (a machine), with queue-capacity constraints between machines. It is assumed that a sufficient supply of raw material is always available at the first queue. Because of the queue-capacity constraints, machine 1 shuts down whenever the inventory buffer fills to capacity, while machine 2 shuts down whenever the buffer empties. In brief, the system consists of three single-server queues in series with queue-capacity constraints and a continuous arrival stream at the first queue.

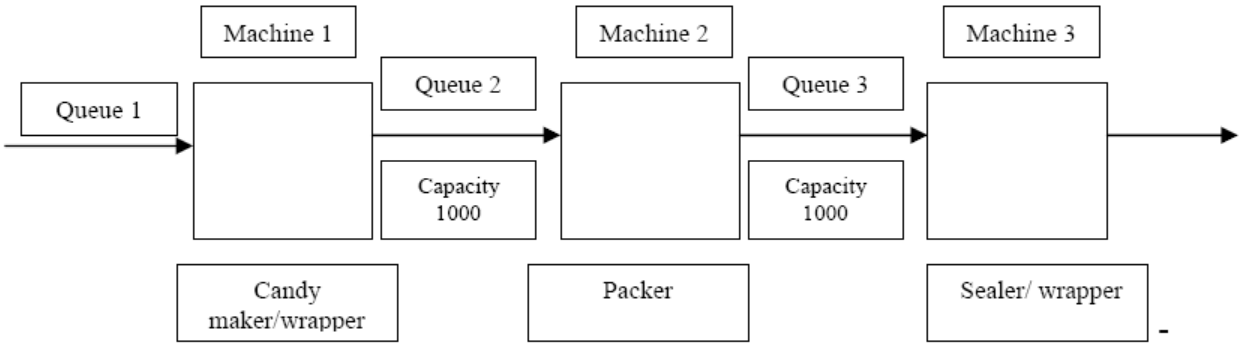


Figure 3: Candy production line

Queueing Notation:-

- Recognizing the diversity of queueing systems, Kendall [1953] proposed a notational system for parallel server systems which has been widely adopted. An abridged version of this convention is based on the format A /B / c / N / K. These letters represent the following system characteristics:
  - ✓ A represents the interarrival time distribution.
  - ✓ B represents the service-time distribution.
  - ✓ [Common symbols for A and B include M (exponential or Markov), D (constant or deterministic), Ek (Erlang of order k), PH (phase-type), H

    (hyperexponential), G (arbitrary or general), and GI (General independent).]

  - ✓ c represents the number of parallel servers.
  - ✓ N represents the system capacity.
  - ✓ K represents the size of the calling population

- For example, M / M / 1 / ∞ / ∞ indicates a single-server system that has unlimited queue capacity and an infinite population of potential arrivals. The interarrival times and service times are exponentially distributed. When N and K are infinite, they may be dropped from the notation. For example, M / M / 1 / ∞ / ∞ is often shortened to M/M/l.

- Additional notation used for parallel server systems is listed in Table 1 given below. The meanings may vary slightly from system to system. All systems will be assumed to have a FIFO queue discipline.

Table 1. Queueing Notation for Parallel Server Systems

P_n	Steady-state probability of having n customers in system
$P_n(t)$	Probability of n customers in system at time t
λ	Arrival rate
λ_e	Effective arrival rate
μ	Service rate of one server
ρ	Server utilization
A_n	Interarrival time between customers n — 1 and n
S_n	Service time of the nth arriving customer
W_n	Total time spent in system by the nth arriving customer
W_n^Q	Total time spent in the waiting line by customer n .
L(t)	The number of customers in system at time /
$L_Q(t)$	The number of customers in queue at time t
L	Long-run time-average number of customers in system
L_Q	Long-run time-average number of customers in queue

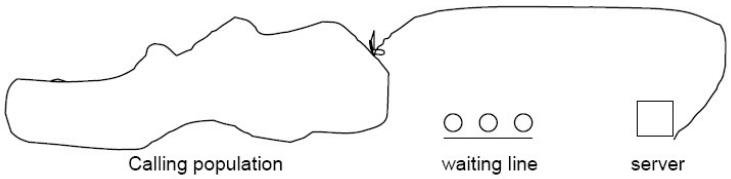
ώ Long-run average time spent in system per customer

$\acute{\omega}_Q$ Long-run average time spent in queue per customer

Simulation of Queuing systems

- A queueing system is described by its calling population, the nature of the arrivals, the service mechanism, the system capacity, and the queueing discipline. A single-channel queueing system is portrayed in figure1.

Figure 1: Queueing System



- In the single-channel queue, the calling population is infinite; that is, if a unit leaves the calling population and joins the waiting line or enters service, there is no change in the arrival rate of other units that may need service.
- Arrivals for service occur one at a time in a random fashion; once they join the waiting line, they are eventually served. In addition, service times are of some random length according to a probability distribution which does not change over time.
- The system capacity; has no limit, meaning that any number of units can wait in line.
- Finally, units are served in the order of their arrival by a single server or channel.
- Arrivals and services are defined by the distributions of the time between arrivals and the distribution of service times, respectively.
- For any simple single or multi-channel queue, the overall effective arrival rate must be less than the total service rate, or the waiting line will grow without bound. When queues grow without bound, they are termed ―explosive‖ or unstable.
- The state of the system is the number of units in the system and the status of the server, busy or idle.
- An event is a set of circumstances that cause an instantaneous change in the state of the system. In a single –channel queueing system there are only two possible events that can affect the state of the system.
  - They are the entry of a unit into the system.
  - The completion of service on a unit.
- The queueing system includes the server, the unit being serviced, and units in the queue. He simulation clock is used to track simulated time. If a unit has just completed service, the simulation proceeds in the manner shown in the flow diagram of figure.2. Note that the server has only two possible states: it is either busy or idle.

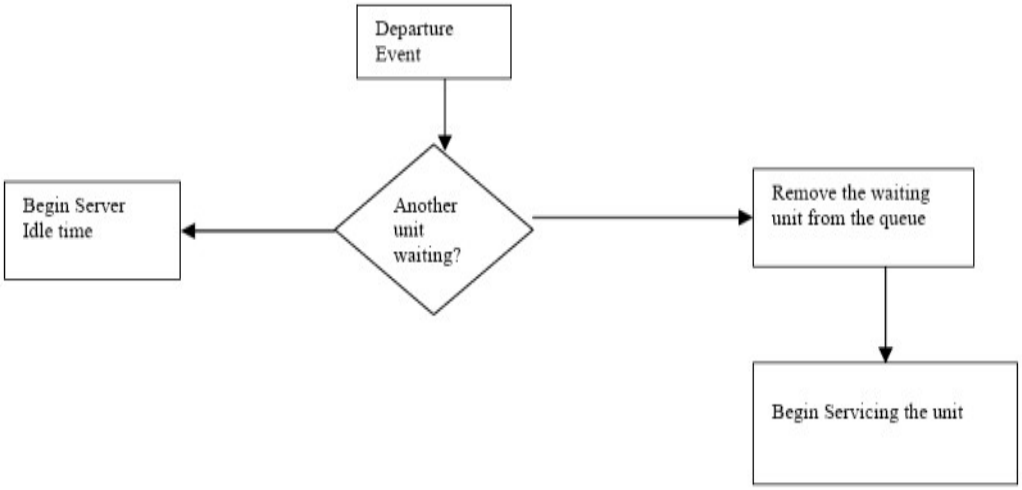


Figure 2: Service-just-completed flow diagram

- The arrival event occurs when a unit enters the system. The flow diagram for the arrival event is shown in figure 3. The unit may find the server either idle or busy; therefore, either the unit begins service immediately, or it enters the queue for the server. The unit follows the course of action shown in fig 4.

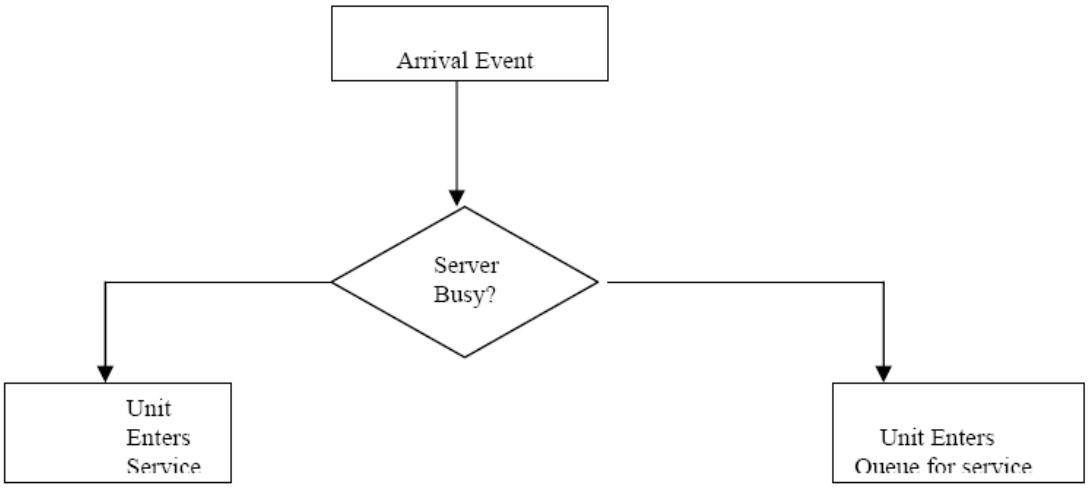


Figure 3: Unit-Entering system flow diagram

- If the server is busy, the unit enters the queue. If the server is idle and the queue is empty, the unit begins service. It is not possible for the server to be idle and the queue to be nonempty.

		Queue Status	
		Not Empty	Empty
Server Status	Busy	Enter Queue	Enter Queue
	Idle	Impossible	Enter Service

Figure 4: Potential unit actions upon arrival

- After the completion of a service the service may become idle or remain busy with the next unit. The relationship of these two outcomes to the status of the queue is shown in fig 5. If the queue is not empty, another unit will enter the server and it will be busy. If the queue is empty, the server will be idle after a service is completed. These two possibilities are shown as the shaded portions of fig 5. It is impossible for the server to become busy if the queue is empty when a service is completed. Similarly, it is impossible for the server to be idle after a service is completed when the queue is not empty.

		Queue Status	
		Not Empty	Empty
Server Status	Busy		Impossible
	Idle	Impossible	

Figure 5: Server outcomes after service completion

Simulation clock times for arrivals and departures are computed in a simulation table customized for each problem. In simulation, events usually occur at random times. In these cases, a statistical model of the data is developed from either data collected and analyzed, or subjective estimates and assumptions.

- Random numbers are distributed uniformly and independently on the interval (0, 1). Random digits are uniformly distributed on the set {0, 1, 2… 9}. Random digits can be used to form random numbers by selecting the proper number of digits for each random number and placing a decimal point to the left of the value selected. The proper number of digits is dictated by the accuracy of the data being used for input purposes. If the input

distribution has values with two decimal places, two digits are taken from a random-digits table and the decimal point is placed to the left to form a random number.

- When numbers are generated using a procedure, they are often referred to as pseudorandom numbers. Since the method is known, it is always possible to know the sequence of numbers that will be generated prior to the simulation.

- In a single-channel queueing system, interarrival times and service times are generated from the distributions of these random variables. The examples that follow show how such times are generated. For simplicity, assume that the times between arrivals were generated by rolling a die five times and recording the up face. Table 1 contains a set of five interarrival times are used to compute the arrival times of six customers at the queuing system.

Table 1: Interarrival and Clock Times

Customer	Interarrival Time	Arrival Time on Clock
1	--	0
2	2	2
3	4	6
4	1	7
5	2	9
6	6	15

- The first customer is assumed to arrive at clock time 0. This starts the clock in operation. The second customer arrives two time units later, at a click time of 2. The third customer arrives four time units later, at a clock time of 6; and so on.

- The second time of interest is the service time. The only possible service times are one, two, three, and four time units. Assuming that all four values are equally likely to occur, these values could have been generated by placing the numbers one through four on chips and drawing the chips from a hat with replacement, being sure to record the numbers selected.

- Now, the interarrival times and service times must be meshed to simulate the singlechannel queueing system. As shown in table 2, the first customer arrives at clock time 0 and immediately begins service, which requires two minutes. Service is completed at clock time 2. The second customer arrives at clock time 2 and is a finished at clock time 3. Note that the fourth customer arrived at clock time 7, but service could not begin until clock time 9. This occurred because customer 3 did not finish service until clock time 9.

- Table 2 was designed specifically for a single-channel queue which serves customers on a first-in, first-out (FIFO) basis. It keeps track of the clock time at which each event

occurs. The second column of table 2 records the clock time of each arrival event, while the last column records the clock time of each departure event.

Table 2: Simulation Table emphasizing Clock Times

A Customer No.	B Arrival Time (Clock)	C Time Service Begins (Clock)	D Service Time (Duration)	E Time Service Ends (Clock)
1	0	0	2	2
2	2	2	1	3
3	6	6	3	9
4	7	9	2	11
5	9	11	1	12
6	15	15	4	19

EXAMPLE 1: Single-Channel Queue

- A small grocery store has only one checkout counter. Customers arrive at this checkout counter at random from 1 to 8 minutes apart. Each possible value of interarrival time has the same probability of occurrence. The service times vary from 1 to 6 min`utes with the probabilities shown in table 5. The problem is to analyze the system by simulating the arrival and service of 20 customers.

Table 5: Service Time Distribution

Service Time (Min)	Probability	Cumulative Frequency	Random Digit Assignment
1	0.10	0.10	01-10
2	0.20	0.30	11-12
3	0.30	0.60	31-60
4	0.25	0.85	61-85
5	0.10	0.95	86-95
6	0.05	1.00	96-00

- A simulation of a grocery store that starts with an empty system is not realistic unless the intention is to model the system from startup or to model until steady state operation is reached.

- A set of uniformly distributed random numbers is needed to generate the arrivals at the checkout counter. Random numbers have the following properties:

1. The set of random numbers is uniformly distributed between 0 and 1.

2. Successive random numbers are independent.

- The time-between-arrival determination is shown in table 6. Note that the first random digits are 913. To obtain the corresponding time between arrivals, enter the fourth column of table 4 and read 8 minutes from the first column of the table. Alternatively, we see that 0.913 is between the cumulative probabilities 0.876 and 1.000, again resulting in 8 minutes as the generated time

Table 6: Time between Arrivals Determination

Customers	Random Digits	Time Between Arrivals (Min)	Customers	Random Digits	Time Between Arrivals (Min)
1	--	--	11	109	1
2	913	8	12	093	1
3	727	6	13	607	5
4	015	1	14	738	6
5	948	8	15	359	3
6	309	3	16	888	8
7	922	8	17	106	1
8	753	.7	18	212	2
9	235	2	19	493	4
10	302	3	20	535	5

- Service times for all 20 customers are shown in table 7. These service times were generated based on the methodology described above, together with the aid of table 5. The first customer‘s service time is 4 minutes because the random digits 84 fall in the bracket 61-85, or alternatively because the derived random number 0.84 falls between the cumulative probabilities 0.61 and 0.85.

Table 7: Service Times Generated

Customer	Random Digits	Service Time (Min)	Customer	Random Digits	Service Time (Min)
1	84	4	11	32	3
2	10	1	12	94	5
3	74	4	13	79	4
4	53	3	14	05	1
5	17	2	15	79	5
6	79	4	16	84	4
7	91	5	17	52	3
8	67	4	18	55	3
9	89	5	19	30	2
10	38	3	20	50	3

- The essence of a manual simulation is the simulation table. These tables are designed for the problem at hand, with columns added to answer the questions posed. The simulation table for the single-channel queue, shown, in table 8 that is an extension of table 2. The first step is to initialize the table by filling in cells for the first customer.
- The first customer is assumed to arrive at time 0. Service begins immediately and finishes at time 4. The customer was in the system for 4 minutes. After the first customer, subsequent rows in the table are based on the random numbers for interarrival time and service time and the completion time of the previous customer. For example, the second customer arrives at time 8. Thus, the server was idle for 4 minutes. Skipping down to the fourth customer, it is seen that this customer arrived at time 15 but could not be served until time 18. This customer had to wait in the queue for 3 minutes. This process continues for all 20 customers.

Table 8: Simulation Table for the queuing problem

A Customers	B Time since last Arrival (Min)	C Arrival Time	D Service Time	E Time Service Begins	F Time customer waits in queue	G Time Service Ends	H Time customer spends in system	I Idle Time of Server
1	--	0	4	0	0	4	4	0
2	8	8	1	8	0	9	1	4
3	6	14	4	14	0	18	4	5
4	1	15	3	18	3	21	6	0
5	8	23	2	23	0	25	2	2
6	3	26	4	26	0	30	4	1
7	8	34	5	34	0	39	5	4
8	7	41	4	41	0	45	4	2
9	2	43	5	45	2	50	7	0
10	3	46	3	50	4	53	7	0
11	1	47	3	53	6	56	9	0
12	1	48	5	56	8	61	13	0
13	5	53	4	61	8	65	12	0
14	6	59	1	65	6	66	7	0
15	3	62	5	66	4	71	9	0
16	8	70	4	71	1	75	5	0
17	1	71	3	75	4	78	7	0
18	2	73	3	78	5	81	8	0
19	4	77	2	81	4	83	6	0
20	5	82	3	83	1	86	4	0
			68		56		124	18

1. The average waiting time for a customer is 2.8minutes. this is determined in the following manner:

$$\text{Average Waiting Time} = \frac{\text{Total time customers wait in queue (min)}}{\text{Total no. of customers}}$$

$$= \frac{56}{20} = 2.8 \text{ minutes}$$

2. The probability that a customer has to wait in the queue is 0.65. This is determined in the following manner:

$$\text{Probability (wait)} = \frac{\text{number of customers who wait}}{\text{Total number of customers}} = \frac{13}{20} = 0.65$$

3. The fraction of idle time of the server is 0.21. This is determined in the following manner:

$$\text{Probability of idle server} = \frac{\text{Total idle time of server (minutes)}}{\text{Total run time of simulation (minutes)}}$$

$$= \frac{18}{86} = 0.21$$

The probability of the server being busy is the complement of 0.21, or 0.79.

4. The average service time is 3.4 minutes, determined as follows:

$$\text{Average service time (minutes)} = \frac{\text{Total service time}}{\text{Total number of customers}}$$

$$= 68/20 = 3.4 \text{ minutes}$$

This result can be compared with the expected service time by finding the mean of the service-time distribution using the equation

$$E(s) = \Sigma\, sp\,(s)$$

Applying the expected-value equation to the distribution in table 2.7 gives an expected service time of:

$$= 1(0.10) + 2(0.20) + 3(0.30) + 4(0.25) + 5(0.10) + 6(0.50)$$

$$= 3.2 \text{ minutes}$$

The expected service time is slightly lower than the average time in the simulation. The longer simulation, the closer the average will be to E (S).

5. The average time between arrivals is 4.3 minutes. This is determined in the following manner:

$$\text{Average time between arrivals (minutes)} = \frac{\text{Sum of all times between arrivals (minutes)}}{\text{Number of arrivals - 1}}$$

$$= \frac{82}{19} = 4.3 \text{ minutes}$$

One is subtracted from the denominator because the first arrival is assumed to occur at time 0. This result can be compared to the expected time between arrivals by finding the mean of the discrete uniform distribution whose endpoints are a = 1 and b = 8. The mean is given by

$$E(A) = \frac{a + b}{2} = \frac{1 + 8}{2} = 4.5 \text{ minutes}$$

The expected time between arrivals is slightly higher than the average. However,

as the simulation becomes longer, the average value of the time between arrivals will approach the theoretical mean, E (A).

6. The average waiting time of those who wait is 4.3 minutes. This is determined in the following manner:

$$\text{Those who wait (minutes)} = \frac{\text{Average waiting time of total time customers wait in queue}}{\text{Total number of customers who wait}}$$

$$= 56/13 = 4.3 \text{ minutes}$$

7. The average time a customer spends in the system is 6.2 minutes. This can be determined in two ways. First, the computation can be achieved by the following relationship:

$$\text{Average time customer Spends in the system} = \frac{\text{total time customers spend in the system}}{\text{Total number of customers}}$$

$$= \frac{124}{20} = 6.2 \text{ minutes}$$

The second way of computing this same result is to realize that the following relationship must hold:

Average time customer spends in the system = average time customer spends waiting in the queue + average time customer spends in service

From findings 1 and 4 this results in:

Average time customer spends in the system = 2.8+ 3.4 = 6.2 minutes.

EXAMPLE 2:- The Able Baker Carhop Problem

- This example illustrates the simulation procedure when there is more than one service channel. Consider a drive-in restaurant where carhops take orders and bring food to the car. Cars arrive in the manner shown in table 1. There are two carhops-Able and Baker. Able is better able to do the job and works a bit faster than Baker. The distribution of their service times are shown in tables 2 and 3.

Table 1: Interarrival distribution of Cars

Time Between arrivals (Min)	Probability	Cumulative Probability	Random Digit Assignment
1	0.25	0.25	01-25
2	0.40	0.65	26-65
3	0.20	0.85	66-85
4	0.15	1.00	86-00

- The simulation proceeds in a manner similar to example 1, except that it its more complex because of the two servers. A simplifying rule is that Able gets the customer if both carhops are idle. Perhaps, Able has seniority. (The solution would be different if the decision were made at random or by any other rule.)

Table 2: Service Distribution of Able

Service Time	Probability	Cumulative Probability	Random-Digit Assignment
2	0.30	0.30	01-30
3	0.28	0.58	31-58
4	0.25	0.83	59-83
5	0.17	1.00	84-00

Table 3: Service Distribution of Baker

Service Time	Probability	Cumulative Probability	Random-Digit Assignment
3	0.35	0.35	01-35
4	0.25	0.60	36-60
5	0.20	0.80	61-80
6	1.00	1.00	81-00

- Here there are more events: a customer arrives, a customer begins service from able, a customer completes service from Able, a customer begins service from Baker, and a customer completes service from Baker. The simulation table is shown in table 4.

- After the first customer, the cells for the other customers must be based on logic and formulas. For example, the ―clock time of arrival‖ in the row for the second customer is computed as follows:

  D2 = D1+C2

- The logic to compute who gets a given customer, and when that service begins, is more complex. The logic goes as follows when a customer arrives: if the customer finds able idle, the customer begins service immediately with able. If able is not idle but baker is, then the customer begins service immediately with baker. If both are busy, the customer begins service with the first server to become free.

The analysis of table 4 results in the following:

1. Over the 62-minute period able was busy 90% of the time.
2. Baker was busy only 69% of the tome. The seniority rule keeps baker less busy.
3. Nine of the 26 arrivals had to wait. The average waiting time for all customers was only about 0.42 minute, which is very small.
4. Those nine who did have to wait only waited an average of 1.22 minutes, which is quite low.

5. In summary, this system seems well balanced. One server cannot handle all the diners, and three servers would probably be too many. Adding an additional server would surely reduce the waiting time to nearly zero. However, the cost of waiting would have to be quite high to justify an additional server.

Table 4: Simulation Table for the Carhop Example

Table 4: Simulation Table for the Carhop Example

A Customer No.	B Random Digits for Arrival	C Time between Arrivals	D Clock time Arrival	E Random Digits for Service	F Time Service Begins	G Service Time	H Time Service Ends	I Time Service Begins	J Service Time	K Time Service Ends	L Time In Queue
1	--	--	0	95	0	5	5				0
2	26	2	2	21				2	3	5	0
3	98	4	6	51	6	3	9				0
4	90	4	10	92	10	5	15				0
5	26	2	12	89				12	6	18	0
6	42	2	14	38	15	3	18				1
7	74	3	17	13	18	2	20				1
8	80	3	20	61	20	4	24				0
9	68	3	23	50				23	4	27	0
10	22	1	24	49	24	3	27				0
11	48	2	26	39	27	3	30				1
12	34	2	28	53				28	4	32	0
13	45	2	30	88	30	5	35				0
14	24	1	31	01				32	3	35	1
15	34	2	33	81	35	4	39				2
16	63	2	35	53				35	4	39	0
17	38	2	37	81	39	4	43				2
18	80	3	40	64				40	5	45	0
19	42	2	42	01	43	2	45				1
20	56	2	44	67	45	4	49				1
21	89	4	48	01				48	3	51	0
22	18	1	49	47	49	3	52				0
23	51	2	51	75				51	5	56	0
24	71	3	54	57	54	3	57				0
25	16	1	55	87				56	6	62	1
26	92	4	59	47	59	3	62		43		0
						56					11

2.4 Simulation of Inventory Systems

An important class of simulation problems involves inventory systems.

- A simple inventory system is shown in fig 1. This inventory system has a periodic review of length N, at which time the inventory level is checked. An order is made to bring the inventory up to the level M. At the end of the review period, an order quantity, Q1, is placed. In this inventory system the lead time is zero. Demand is shown as being uniform over the time period in fig 1. In actuality, demands are not usually uniform and do fluctuate over time. One possibility is that demands all occur at the beginning of the cycle. Another is that the lead time is random of some positive length.

- Notice that in the second cycle, the amount in inventory drops below zero, indicating a shortage. In fig 1, these units are backordered. When the order arrives, the demand for the backordered items is satisfied first. To avoid shortages, a buffer, or safety, stock would need to be carried.

- ❖ Carrying stock in inventory has an associated cost attributed to the interest paid on the funds borrowed to buy the items. Other costs can be placed in the carrying or holding cost column: renting of storage space, hiring guards, and so on.
- ❖ An alternative to carrying high inventory is to make more frequent reviews, and consequently, more frequent purchases or replenishments. This has an associated cost: the ordering cost. Also, there is a cost in being short. Larger inventories decrease the possibilities of shortages. These costs must be traded off in order to minimize the total cost of an inventory system.

- ❖ The total cost of an inventory system is the measure of performance. This can be affected by the policy alternatives. For example, in fig 1, the decision maker can control the maximum inventory level, M, and the cycle, N.

- ❖ In an (M, N) inventory system, the events that may occur are: the demand for items in the inventory, the review of the inventory position, and the receipt of an order at the end of each review period. When the lead time is zero, as in fig 1, the last two events occur simultaneously.

    - The Newspaper Seller‘s Problem

        - A classical inventory problem concerns the purchase and sale of newspapers. The paper seller buys the papers for 33 cents each and sells them for 50 cents each. Newspapers not sold at the end of the day are sold as scrap for 5 cents each. Newspapers can be purchased in bundles of 10. Thus, the paper seller can buy 50, 60, and so on.
        - There are three types of Newsday‘s, ―good,‖ ―fair,‖ and ―poor,‖ with probabilities of 0.35, 0.45, and 0.25, respectively. The distribution of papers demanded on each of these days is given in table 2.15. The problem is to determine the optimal number of papers the newspaper seller should purchase. This will be accomplished by simulating demands for 20 days and recording profits from sales each day.

The profits are given by the following relationship:

$$\text{Profit} = \left[\text{Revenue form sales} - \text{cost of newspapers} - \text{lost profit form excess demand} + \text{salvage from sale of scrap papers}\right]$$

Table 5: Distribution of Newspaper Demanded

	Demand Probability Distribution		
Demand	Good	Fair	Poor
40	0.03	0.10	0.44
50	0.05	0.18	0.22
60	0.15	0.40	0.16

- Form the problem statement, the revenue from sales is 50 cents for each paper sold. The Cost 9 of newspapers is 33 cents for each paper purchased. The lost profit from excess demand is 17 cents for each paper

demanded that could not be provided. Such a shortage cost is somewhat controversial but makes the problem much more interesting. The salvage value of scrap papers is 5 cents each.

- Tables 6 and 7 provide the random-digit assignments for the types of Newsday‘s and the demands for those Newsday‘s.

Table 6: Random Digit Assignment for Type of Newsday

Type of Newsday	Probability	Cumulative probability	Random Digit Assignment
Good	0.35	0.35	01 – 35
Fair	0.45	0.80	36 – 80
Poor	0.20	1.00	81 - 00

Table 7: Random Digit Assignment for Newspapers Demanded

	Cumulative Distribution			Random Digit Assignment		
Demand	Good	Fair	Poor	Good	Fair	Poor
40	0.03	0.10	0.44	01 – 03	01 – 10	01 – 44
50	0.08	0.28	0.66	04- 08	11 – 28	45 – 66
60	0.23	0.68	0.82	09 - 23	29 - 68	67 - 82

- The simulation table for the decision to purchase 70 newspapers is shown in table 8.On day 1 the demand is for 60 newspapers. The revenue from the sale of 60 newspapers is $30.00. Ten newspapers are left over at the end of the day. The salvage value at 5 cents each is 50 cents. The profit for the first day is determined as follows:

  Profit = $30.00 - $ 23.10 – 0 + $.50 = $7.40

Table 8: Simulation table fro purchase of 70 newspapers

Day	Random digits for type of Newsday	Type of Newsday	Random digits for demand	Demand	Revenue from sales	Lost profit from excess demand	Salvage from sale of scrap	Daily profit
1	94	Poor	80	60	$30	-	$0.50	$7.40
2	77	Fair	20	50	$25	-	$1.0	$2.90
3	49	Fair	15	50	$25	-	$1.0	$2.90

- On the fifth day the demand is greater than the supply. The revenue from sales is $35.00, since only 70 papers are available under this policy. An additional 20 papers could have been sold. Thus, a lost profit of $3.40 (20*17 cents) is assessed. The daily profit is determined as follows:

  Profit = $35.00 - $23.10 - $3.40 + 0 = $8.50

- The profit for the 20-day period is the sum of the daily profits, $174.90. it can also be computed from the totals for the 20 days of the simulation as follows:
  Total profit = $645 - $462 - $13 .60 + $5.50 = $174.90

- In general, since the results of one day are independent of those of previous days, inventory problems of this type are easier than queueing problems.

Simulation of an (M, N) Inventory System

- Suppose that the maximum inventory level, M, is 11 units and the review period, N, is 5 days. The problem is to estimate, by simulation, the average ending units in inventory and the number of days when a shortage condition occurs. The distribution of the number of units demanded per day is shown in table 9. In this example, lead-time is a random variable, as shown in table 10. Assume that orders are placed at the close of business and are received for inventory at the beginning as determined by the lead-time.

Table 9: Random digits assignments for daily demand

Demand	Probability	Cumulative Probability	Random digits assignments
0	0.10	0.10	01 – 10
1	0.25	0.35	11 – 35
2	0.35	0.70	36 - 70

Table 10: Random digit assignments for lead time

Lead Time (Days)	Probability	Cumulative Probability	Random digits assignments
1	0.6	0.6	1 – 6
2	0.3	0.9	7 – 9
3	0.1	1.0	0

Table 11: Simulation table for (M, N) Inventory System

Cycle	Day	Beginning Inventory	Random digits for demand	Demand	Ending Inventory	Shortage quantity	Order quantity	Random digits For lead time	Days order
1	1	3	24	1	2	0	-	-	
	2	2	35	1	1	0	-	-	
	3	9	65	2	7	0	-	-	
	4	7	81	3	4	0	-	-	
	5	4	54	2	2	0	9	5	

Note : Refer cycle 2.3.4.5 from Text book page no 47.

- To make an estimate of the mean units in ending inventory, many cycles would have to be simulated. For purposes of this example, only five cycles will be shown. The reader is asked to continue the example as an exercise at the end of the chapter.

- The random-digit assignments for daily demand and lead time are shown in the rightmost columns of tables 9 and 10. The resulting simulation table is shown in table 11.The simulation has been started with the inventory level at 3 units and an order of 8 units scheduled to arrive in 2 days time.

- Following the simulation table for several selected days indicates how the process operates. The order for 8 units is available on the morning of the third day of the first cycle, raising the inventory level from 1 unit to 9 units; demands during the remainder of the first cycle reduced the ending inventory level to 2 units on the fifth day. Thus, an order for 9 units was placed. The lead time for this order was 1 day. The order of 9 units was added to inventory on the morning of day 2 of cycle 2.

- Notice that the beginning inventory on the second day of the third cycle was zero. An order for 2 units on that day led to a shortage condition. The units were backordered on that day and the next day;; also on the morning of day 4 of cycle 3 there was a beginning inventory of 9 units that were backordered and the 1 unit demanded that day reduced the ending inventory to 4 units.

- Based on five cycles of simulation, the average ending inventory is approximately 3.5 (88/25) units. On 2 of 25 days a shortage condition existed.

UNIT 2

GENERAL PRINCIPLES

Introduction

- This chapter develops a common framework for the modeling of complex systems using discrete-event simulation.
- It covers the basic building blocks of all discrete-event simulation models: entities and attributes, activities and events.
- In discrete-event simulation, a system is modeled in terms of its state at each point in time; the entities that pass through the system and the entities that rep-j resent system resources; and the activities and events that cause system state to change.
- This chapter deals exclusively with dynamic, stochastic system (i.e. involving time and containing random elements) which changes in a discrete manner.

2.1 Concepts in Discrete-Event Simulation

- The concept of a system and a model of a system were discussed briefly in earlier chapters.
- This section expands on these concepts and develops a framework for the development of a discrete-event model of a system.
- The major concepts are briefly defined and then illustrated with examples:
- **System:** A collection of entities (e.g., people and machines) that ii together over time to accomplish one or more goals.
- **Model:** An abstract representation of a system, usually containing structural, logical, or mathematical relationships which describe a system in terms of state, entities and their attributes, sets, processes, events, activities, and delays.

- **System state:** A collection of variables that contain all the information necessary to describe the system at any time.

- **Entity:** Any object or component in the system which requires explicit representation in the model (e.g., a server, a customer, a machine).

- **Attributes:** The properties of a given entity (e.g., the priority of a v customer, the routing of a job through a job shop).

- **List:** A collection of (permanently or temporarily) associated entities ordered in some logical fashion (such as all customers currently in a waiting line, ordered by first come, first served, or by priority).

- **Event:** An instantaneous occurrence that changes the state of a system as an arrival of a new customer).

- **Event notice:** A record of an event to occur at the current or some future time, along with any associated data necessary to execute the event; at a minimum, the record includes the event type and the event time.

- **Event list:** A list of event notices for future events, ordered by time of occurrence; also known as the future event list (FEL).

- **Activity:** A duration of time of specified length (e.g., a service time or arrival time), which is known when it begins (although it may be defined in terms of a statistical distribution).

- **Delay:** A duration of time of unspecified indefinite length, which is not known until it ends (e.g., a customer's delay in a last-in, first-out waiting line which, when it begins, depends on future arrivals).

- **Clock:** A variable representing simulated time. · The future event list is ranked by the event time recorded in the event notice.

❖ An activity typically represents a service time, an interarrival time, or any other processing time whose duration has been characterized and defined by the modeler.
❖ An activity‘s duration may be specified in a number of ways:

1. Deterministic-for example, always exactly 5 minutes.

2. Statistical-for example, as a random draw from among 2,5,7 with equal probabilities.

3. A function depending on system variables and/or entity attributes.

- The duration of an activity is computable from its specification at the instant it begins.

- A delay's duration is not specified by the modeler ahead of time, but rather is determined by system conditions.

- A delay is sometimes called a conditional wait, while an activity is called unconditional wait.

- The completion of an activity is an event, often called primary event.

- The completion of a delay is sometimes called a conditional or secondary event.

EXAMPLE 3.1 (Able and Baker, Revisited)

Consider the Able-Baker carhop system of Example 2.2. A discrete- event model has the following components:

System State

LQ(t), the number of cars waiting to be served at time t

LA(t), 0 or 1 to indicate Able being idle or busy at time t

LB (t), 0 or 1 to indicate Baker being idle or busy at time t

Entities

Neither the customers (i.e., cars) nor the servers need to be explicitly represented, except in terms of the state variables, unless certain customer averages are desired (compare Examples 3.4 and 3.5)

Events

Arrival event

Service completion by Able

Service completion by Baker

Activities

Interarrival time, defined in Table 2.11

Service time by Able, defined in Table 2.12

Service time by Baker, defined in Table 2.13

Delay

A customer's wait in queue until Able or Baker becomes free.

2.2 The Event-Scheduling/Time-Advance Algorithm

- The mechanism for advancing simulation time and guaranteeing that all events occur in correct chronological order is based on the future event list (FEL).
- This list contains all event notices for events that have been scheduled to occur at a future time.
- At any given time t, the FEL contains all previously scheduled future events and their associated event times
- The FEL is ordered by event time, meaning that the events are arranged chronologically; that is, the event times satisfy

$t < t1 <= t2 <= t3 <= \ldots,<= tn$

t is the value of CLOCK, the current value of simulated time. The event dated with time t1 is called the imminent event; that is, it is the next event will occur. After the system snapshot at simulation time

CLOCK == t has been updated, the CLOCK is advanced to simulation time

CLOCK _= t1 and the imminent event notice is removed from the FEL

and the event executed.. This process repeats until the simulation is over.

- The sequence of actions which a simulator must perform to advance the clock system snapshot is called the event-scheduling/time-advance algorithm

Old system snapshot at time t

CIK	System State		Future Event List
T	(5,1,6)		(3, t1)— Type 3 event to occur at time *t1* (1, t2)— Type 1 event to occur at time *t2* (1, t3)- Type 1 event to occur at time t_3 (2, *tn*)— Type 2 event to occur at time t_n

Event-scheduling/time-advance algorithm

Step 1. Remove the event notice for the imminent event

(event 3, time t\) from PEL

Step 2. Advance CLOCK to imminent event time

(i.e., advance CLOCK from r to t1).

Step 3. Execute imminent event: update system state,

change entity attributes, and set membership as needed.

Step 4. Generate future events (if necessary) and

place their event notices on PEL ranked by event time.

(Example: Event 4 to occur at time t*, where t2 < t* < t3.)

Step 5. Update cumulative statistics and counters.

New system snapshot at time t1

OCK	System State		Future Event List
t1	(5,1,5)		(1, t2)— Type 1 event to occur at time t1 (4, t*)— Type 4 event to occur at time t* (1, t_3)— Type 1 event to occur at time t_3 (2, tn)— Type 2 event to occur at time tn

Figure 3.2 Advancing simulation time and updating system image

- The management of a list is called list processing
- The major list processing operations performed on a FEL are removal of the imminent event, addition of a new event to the list, and occasionally removal of some event (called cancellation of an event).
- As the imminent event is usually at the top of the list, its removal is as efficient as possible. Addition of a new event (and cancellation of an old event) requires a search of the list. The removal and addition of events from the PEL is illustrated in Figure 3.2.

  - When event 4 (say, an arrival event) with event time t* is generated at step 4, one possible way to determine its correct position on the FEL is to conduct a top-down search:
    If t* < t2, place event 4 at the top of the FEL.

    If t2 < t* < t3, place event 4 second on the list.

    If t3, < t* < t4, place event 4 third on the list.

    If tn < t*, event 4 last on the list.

  - Another way is to conduct a bottom-up search.
- The system snapshot at time 0 is defined by the initial conditions and the generation of the so-called exogenous events.

- ❖ The method of generating an external arrival stream, called bootstrapping.

- ❖ Every simulation must have a stopping event, here called E, which defines how long the simulation will run. There are generally two ways to stop a simulation:

    1. At time 0, schedule a stop simulation event at a specified future time TE. Thus, before simulating, it is known that the simulation will run over the time interval [0, TE]. Example: Simulate a job shop for TE = 40 hours.

    2. Run length TE is determined by the simulation itself. Generally, TE is the time of occurrence of some specified event E. Examples: TE is the time of the 100th service completion at a certain service center. TE is the time of breakdown of a complex system.

2.3 ***World Views***

- When using a simulation package or even when using a manual simulation, a modeler adopts a world view or orientation for developing a model.
- Those most prevalent are the event scheduling world view, the process-interaction worldview, and the activity-scanning world view.
- When using a package that supports the process-interaction approach, a simulation analyst thinks in terms of processes .
- When using the event-scheduling approach, a simulation analyst concentrates on events and their effect on system state.
- The process-interaction approach is popular because of its intuitive appeal, and because the simulation packages that implement it allow an analyst to describe the process flow in terms of high-level block or network constructs.
- Both the event-scheduling and the process-interaction approaches use a / variable time advance.
- The activity-scanning approach uses a fixed time increment and a rule-based approach to decide whether any activities can begin at each point in simulated time.
- The pure activity scanning approach has been modified by what is called the three-phase approach.
- In the three-phase approach, events are considered to be activity duration-zero time units. With this definition, activities are divided into two categories called B and C.

    - ✓ B activities: Activities bound to occur; all primary

- ✓ C activities: Activities or events that are conditional upon certain conditions being true.

- With the. three-phase approach the simulation proceeds with repeated execution of the three phases until it is completed:
  - ➢ Phase A: Remove the imminent event from the FEL and advance the clock to its event time. Remove any other events from the FEL that have the event time.
  - ➢ Phase B: Execute all B-type events that were removed from the FEL.
  - ➢ Phase C: Scan the conditions that trigger each C-type activity and activate any whose conditions are met. Rescan until no additional C-type activities can begin or events occur.
- The three-phase approach improves the execution efficiency of the activity scanning method.

EXAMPLE 3.2 (Able and Baker, Back Again)

Using the three-phase approach, the conditions for beginning each activity in Phase C are:

Activity	*Condition*
Service time by Able	A customer is in queue and Able is idle,
Service time by Baker	A customer is in queue, Baker is idle, and Able is busy.

2.5**Manual Simulation Using Event Scheduling**

- In an event-scheduling simulation, a simulation table is used to record the successive system snapshots as time advances.
- Lets consider the example of a grocery shop which has only one checkout counter.

Example 3.3 (Single-Channel Queue)

- The system consists of those customers in the waiting line plus the one (if any) checking out.
- The model has the following components:

  - **System state** *(LQ(t), L*S(t)), where *LQ(t)* is the number of customers in the waiting line, and *LS(t)* is the number being served (0 or 1) at time *t.*

  - **Entities** The server and customers are not explicitly modeled, except in terms of the state variables above.

  - **Events**
    Arrival (A)

    Departure *(D)*

    Stopping event (£"), scheduled to occur at time 60.

  - **Event notices**
    (A, i). Representing an arrival event to occur at future time *t*

    (D, t), representing a customer departure at future time t

    (£, 60), representing the simulation-stop event at future time 60

  - **Activities**
    Inter arrival time, denned in Table 2.6

    Service time, defined in Table 2.7

  - **Delay**
    Customer time spent in waiting line.

- In this model, the FEL will always contain either two or three event notices.

- The effect *of* the arrival and departure events was first shown in Figures below

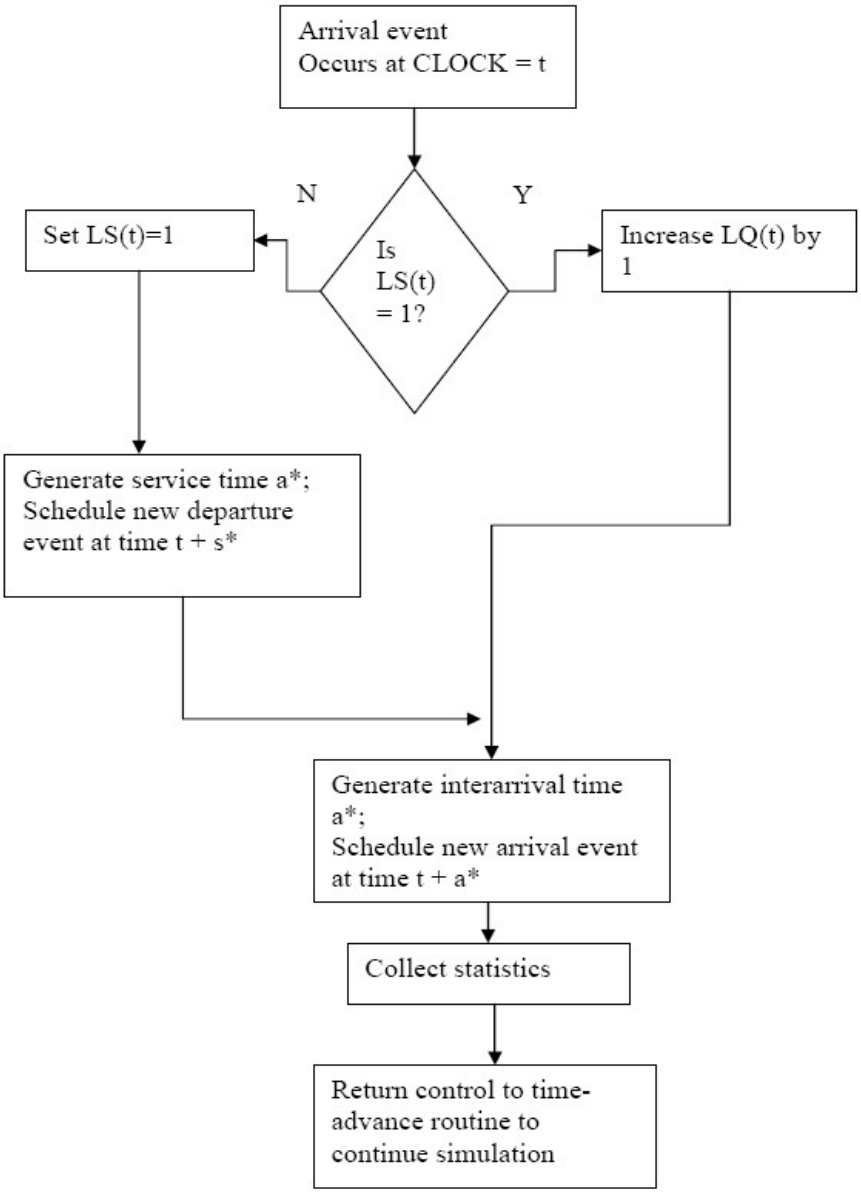


Fig 1(A): Execution of the arrival event.

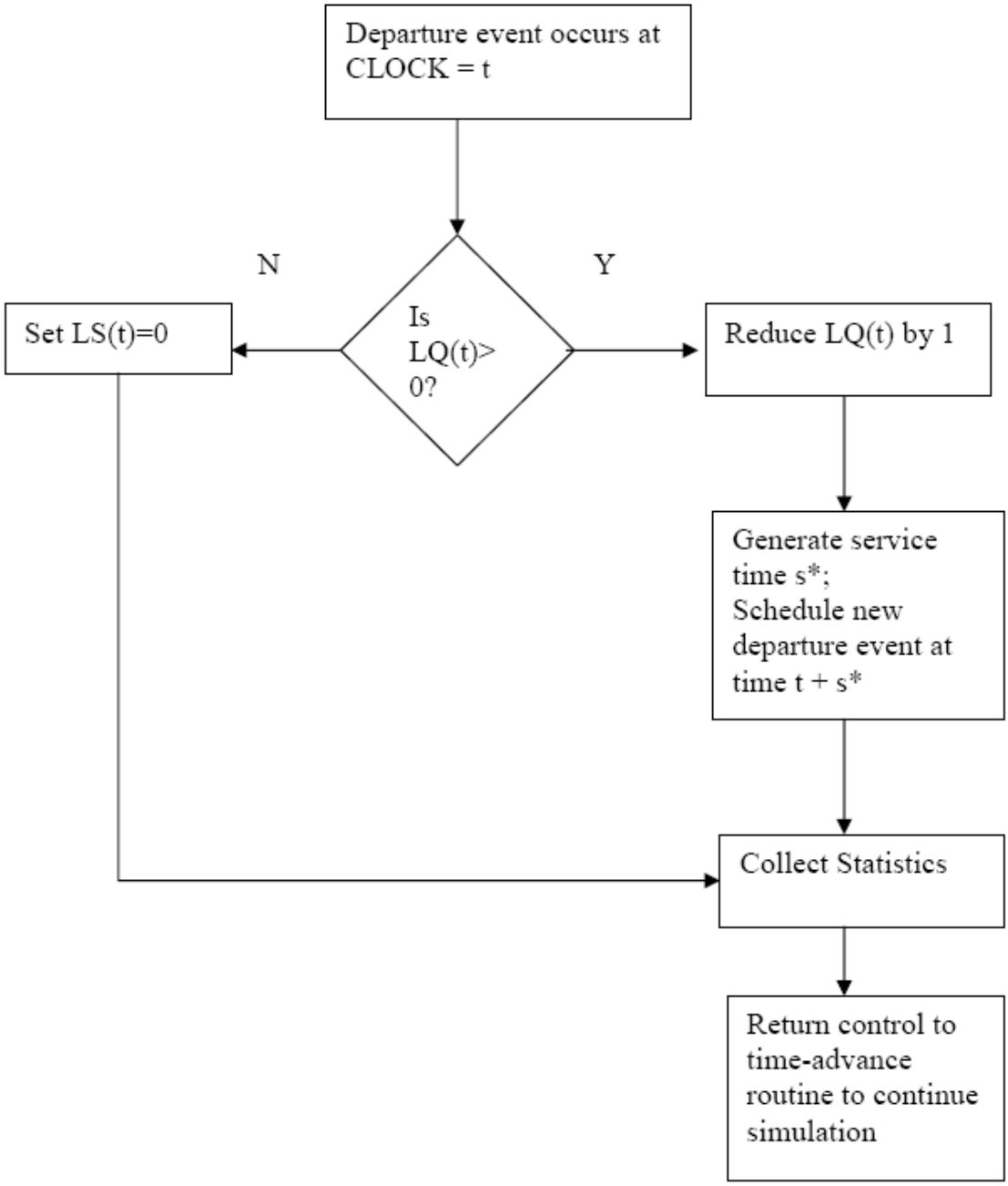


Fig 1(B): Execution of the departure event.

- Initial conditions are that the first customer arrives at time 0 and begins service.
- This is reflected in Table below by the system snapshot at time zero (CLOCK = 0), with LQ (0) = 0, LS (0) = 1, and both a departure event

  and arrival event on the FEL.

- The simulation is scheduled to stop at time 60.
- Two statistics, server utilization and maximum queue length, will be collected.
- Server utilization is defined by total server busy time .(B) divided by total time*(Te)*.

- Total busy time, B, and maximum queue length MQ, will be accumulated as the simulation progresses.
- As soon as the system snapshot at time CLOCK = 0 is complete, the simulation begins.
- At time 0, the imminent event is (D, 4).
- The CLOCK is advanced to time 4, and (D, 4) is removed from the FEL.
- Since *LS(t)* = 1 for 0 <= t <= 4 (i.e., the server was busy for 4 minutes), the cumulative busy time is Increased from B = 0 to B = 4.
- By the event logic in Figure 1(B), set LS (4) = 0 (the server becomes idle).
- The FEL is left with only two future events, (A, 8) and (E, 0).
- The simulation CLOCK is next advanced to time 8 and an arrival event is executed.
- The simulation table covers interval [0,9].

Simulation table for checkout counter.

Clock	LQ(t)	LS(t)	FEL	Comment	B	MQ
0	0	1	(D,4) (A,8) (E,60)	First A occurs (a* = 8) schedule next A (s* = 4) schedule next D	0	0
4	0	0	(A,8) (E,60)	First D occurs;(D,4)	4	0
8	0	1	(D,9) (A,14) (E,60)	Second A occurs;(A,8) (a* = 6) schedule next A (s* = 1) schedule next D	4	0
9	0	0	(A,14) (E,60)	Second D occurs;(D,9)	5	0

Example 3.4 (The Checkout-Counter Simulation, Continued)

- Suppose the system analyst desires to estimate the mean response time and mean proportion of customers who spend 4 or more minutes in the system the above mentioned model has to be modified.

  - **Entities** (Ci, t), representing customer Ci who arrived at time t.

  - **Event notices** *(A, t,* Ci), the arrival of customer Ci at future time t
    (D, f, Cj), the departure of customer Cj at future time t.

  - **Set** "CHECKOUT LINE," the set of all customers currently at the checkout Counter (being served or waiting to be served), ordered by time of arrival

- Three new statistics are collected: S, the sum of customers response times for all customers who have departed by the current time; F, the total number of customers who spend 4 or more minutes at the check out counter; ND the total number of departures up to the current simulation time.

- These three cumulative statistics are updated whenever the departure event occurs.

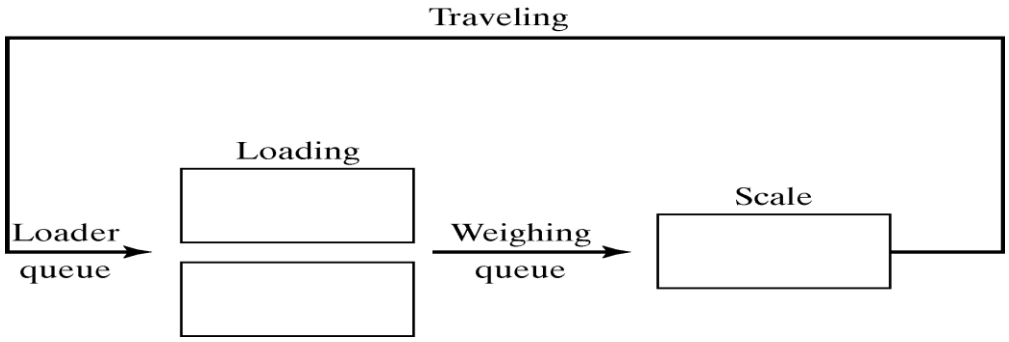
- The simulation table is given below

Simulation Table for Example 3.4

System state					Cumulative statistics		
Clock	LQ(t)	LS(t)	Checkout line	FEL	S	N_D	F
0	0	1	(C 1,0)	(D,4,C1) (A,8,C2) (E,60)	0	0	0
4	0	0		(A,8,C2) (E,60)	4	1	1
8	0	0	(C2,8)	(D,9,C2) (A,14,C3) (E,60)	4	1	1
9	0	0		(A,14,C3) (E,60)	5	2	1

Example 3.5 (The Dump Truck Problem)

- Six dump trucks are used to haul coal from the entrance of a small mine to the railroad. Each truck is loaded by one of two loaders. After loading, a truck immediately moves to scale, to be weighted as soon as possible. Both the loaders and the scale have a first come, first serve waiting line(or queue) for trucks.The time taken to travel from loader to scale is considered negligible.After being weighted, a truck begins a travel time and then afterward returns to the loader queue.



Distribution of Loading for the Dump Truck

Loading time	Probability	Cumulative probability	Random-Digit Assignment
5	0.30	0.30	1-3
10	0.50	0.80	4-8
15	0.20	1.00	9-0

Distribution of Weighing Time for the Dump Truck

Weighing time	Probability	Cumulative probability	Random-Digit Assignment
12	0.70	0.70	1-7
16	0.30	1.00	8-0

Distribution of Travel Time for the Dump Truck

Travel time	Probability	Cumulative probability	Random-Digit Assignment
40	0.40	0.40	1-4
60	0.30	0.70	5-7
80	0.20	0.90	8-9
100	0.10	1.00	0

- The model has the following components:

  - **System state**
    [LQ(t), L(t),WQ(r), W(r)], where

    LQ(t) = number of trucks in loader queue

    L(t) = number of trucks (0,1, or 2)being Loaded

    WQ(t*)*= number of trucks in weigh queue

    W(t) = number of trucks (0 or 1) being weighed, all atsimulation time *t*

  - **Event notices**
    (ALQ, t, DTi), dump truck arrives at loader queue (ALQ) at time t

    (EL, t, *DTi),* dump truck i ends loading *(EL)* at time t

    (EW, t, *DTi),* dump truck i ends weighing *(EW)* at time t

  - **Entities** The six dump trucks *(DTI,..., DT6)*

  - **Lists**
    Loader queue, all trucks waiting to begin loading, ordered on a first-come, first-served basis

    Weigh queue, all trucks waiting to be weighed, ordered on a first-come, first-serve basis.

  - **Activities** Loading time, weighing time, and travel time.

  - **Delays** Delay at loader queue, and delay at scale.

- The activity times are taken from the following list

Loading time	10	5	5	10	15	10	10
Weighing time	12	12	12	16	12	16	
Travel time	60	100	40	40	80		

- **Simulation table for Dump Truck problem**

System state **Lists**

cumulative stat

Clock t	LQ(t)	L(t)	WQ(t)	W(t)	Loader queue	Weigh queue	FEL	B_L	B_S
0	3	2	0	1	DT4 DT5 DT6		(EL,5,DT3) (EL,10,DT2) (EL,12,DT1)	0	0
5	2	2	1	1	DT5 DT6	DT3	(EL,10,DT2) (EL,5 + 5 ,DT4) (EW,12,DT1)	10	5
10	1	2	2	1	DT6	DT3 DT2	(EL,10,DT4) (EW,12,DT1) (EL,10+10,DT5)	20	10
10	0	2	3	1		DT3 DT2 DT4	(EW,12,DT1) (EL,20,DT5) (EL,10+15,DT6)	20	10
12	0	2	2	1		DT2 DT4	(EL,20,DT5) (EW,12+12,DT3) (EL,25,DT6) (ALQ,12+60,DT1)	24	12
20	0	1	3	1		DT2 DT4 DT5	(EW,24,DT3) (EL,25,DT6) (ALQ,72,DT1)	40	20
24	0	1	2	1		DT4 DT5	(EL,25,DT6) (EW,24+12,DT2) (ALQ,72,DT1) (ALQ,24+100,DT3)	44	24

Average Loader Utilization = 44 / 2 = 0.92

24

Average Scale Utilization =24/24 = 1.00

UNIT-3 **STATISTICAL MODELS IN SIMULATION**

Purpose & Overview

- The world the model-builder sees is probabilistic rather than deterministic.
  - Some statistical model might well describe the variations.
- An appropriate model can be developed by sampling the phenomenon of interest:
  - Select a known distribution through educated guesses
  - Make estimate of the parameter(s)
  - Test for goodness of fit
- In this chapter:
  - Review several important probability distributions
  - Present some typical application of these models

3.1 Review of Terminology and Concepts

- In this section, we will review the following concepts:
  - Discrete random variables
  - Continuous random variables
  - Cumulative distribution function
  - Expectation

Discrete Random Variables [Probability Review]

- *X* is a discrete random variable if the number of possible values of *X* is finite, or countably infinite.
- Example: Consider jobs arriving at a job shop.
  - Let *X* be the number of jobs arriving each week at a job shop.
  - R_x = possible values of *X* (range space of *X*) = *{0,1,2,…}*
  - $p(x_i)$ = probability the random variable is $x_i = P(X = x_i)$
  - $p(x_i)$, $i = 1,2, \ldots$ must satisfy:

    1. $p(x_i) \geq 0, \text{ for all } i$
    2. $\sum_{i=1}^{\infty} p(x_i) = 1$

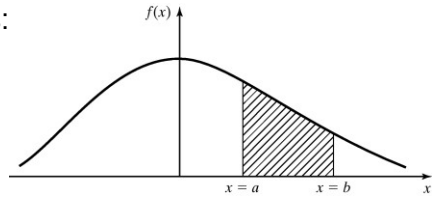
  - The collection of pairs $[x_i, p(x_i)]$, $i = 1,2,\ldots,$ is called the probability distribution of *X*, and $p(x_i)$ is called the probability mass function (pmf) of *X*.

Continuous Random Variables [Probability Review]

- *X* is a continuous random variable if its range space R_x is an interval or a collection of intervals.
- The probability that *X* lies in the interval *[a,b]* is given by:

$$P(a \le X \le b) = \int_a^b f(x)dx$$

- *f(x)*, denoted as the pdf of *X*, satisfies:
  1. $f(x) \ge 0$, for all x in R_X
  2. $\int_{R_X} f(x)dx = 1$
  3. $f(x) = 0$, if x is not in R_X

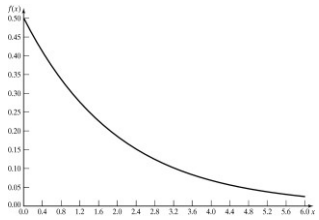


- Properties
  1. $P(X = x_0) = 0$, because $\int_{x_0}^{x_0} f(x)dx = 0$
  2. $P(a \le X \le b) = P(a \prec X \le b) = P(a \le X \prec b) = P(a \prec X \prec b)$

Continuous Random Variables [Probability Review]

- Example: Life of an inspection device is given by *X*, a continuous random variable with pdf:

$$f(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$



  - *X* has an exponential distribution with mean 2 years
  - Probability that the device's life is between 2 and 3 years is:

$$P(2 \le x \le 3) = \frac{1}{2}\int_2^3 e^{-x/2}dx = 0.14$$

Cumulative Distribution Function [Probability Review]

- Cumulative Distribution Function (cdf) is denoted by *F(x)*, where *F(x) = P(X <= x)*
  - If *X* is discrete, then $F(x) = \sum_{\substack{\text{all} \\ x_i \le x}} p(x_i)$
  - If *X* is continuous, then $F(x) = \int_{-\infty}^{x} f(t)dt$
- Properties
  1. F is nondecreasing function. If $a \prec b$, then $F(a) \le F(b)$
  2. $\lim_{x \to \infty} F(x) = 1$
  3. $\lim_{x \to -\infty} F(x) = 0$
- All probability question about *X* can be answered in terms of the cdf, e.g.:

$$P(a \prec X \le b) = F(b) - F(a), \text{ for all } a \prec b$$

Cumulative Distribution Function [Probability Review]

- Example: An inspection device has cdf:

$$F(x) = \frac{1}{2}\int_0^x e^{-t/2}dt = 1 - e^{-x/2}$$

  - The probability that the device lasts for less than 2 years:

$$P(0 \le X \le 2) = F(2) - F(0) = F(2) = 1 - e^{-1} = 0.632$$

  - The probability that it lasts between 2 and 3 years:

$$P(2 \le X \le 3) = F(3) - F(2) = (1 - e^{-(3/2)}) - (1 - e^{-1}) = 0.145$$

Expectation [Probability Review]

- The expected value of *X* is denoted by *E(X)*
  - If *X* is discrete $E(x) = \sum_{\text{all } i}^{\infty} x_i p(x_i)$
  - If *X* is continuous $E(x) = \int_{-\infty}^{\infty} x f(x) dx$
  - a.k.a the mean, m, or the 1st moment of *X*
  - A measure of the central tendency
- The variance of *X* is denoted by *V(X)* or *var(X)* or σ^2
  - Definition: $V(X) = E[(X - E[X]^2]$
  - Also, $V(X) = E(X^2) - [E(x)]^2$
  - A measure of the spread or variation of the possible values of X around the mean
    - The standard deviation of *X* is denoted by σ
  - Definition: square root of *V(X)*
  - Expressed in the same units as the mean

Expectations

[Probability Review]

- Example: The mean of life of the previous inspection device is:

$$E(X) = \frac{1}{2}\int_0^\infty x e^{-x/2} dx = -xe^{-x/2}\Big|_0^\infty + \int_0^\infty e^{-x/2} dx = 2$$

- To compute variance of X, we first compute $E(X^2)$:

$$E(X^2) = \frac{1}{2}\int_0^\infty x^2 e^{-x/2} dx = -x^2 e^{-x/2}\Big|_0^\infty + \int_0^\infty e^{-x/2} dx = 8$$

0 0 0

- Hence, the variance and standard deviation of the device's life are:

$$V(X) = 8 - 2^2 = 4$$
$$\sigma = \sqrt{V(X)} = 2$$

3.2 Useful Statistical Models

- In this section, statistical models appropriate to some application areas are presented. The areas include:
  - Queueing systems
  - Inventory and supply-chain systems
  - Reliability and maintainability
  - Limited data

Queueing Systems [Useful Models]

- In a queueing system, interarrival and service-time patterns can be probablistic (for more queueing examples, see Chapter 2).
- Sample statistical models for interarrival or service time distribution:
  - Exponential distribution: if service times are completely random
  - Normal distribution: fairly constant but with some random variability (either positive or negative)
  - Truncated normal distribution: similar to normal distribution but with restricted value.
  - Gamma and Weibull distribution: more general than exponential (involving location of the modes of pdf's and the shapes of tails.)

Inventory and supply chain [Useful Models]

- In realistic inventory and supply-chain systems, there are at least three random variables:
  - The number of units demanded per order or per time period
  - The time between demands
  - The lead time
- Sample statistical models for lead time distribution:
  - Gamma
- Sample statistical models for demand distribution:
  - Poisson: simple and extensively tabulated.
  - Negative binomial distribution: longer tail than Poisson (more large demands).
  - Geometric: special case of negative binomial given at least one demand has occurred.

- Time to failure (TTF)
  - Exponential: failures are random
  - Gamma: for standby redundancy where each component has an exponential TTF
  - Weibull: failure is due to the most serious of a large number of defects in a system of components
  - Normal: failures are due to wear
  - For cases with limited data, some useful distributions are:
    - Uniform, triangular and beta
  - Other distribution: Bernoulli, binomial and hyper exponential.

3.3 Discrete Distributions

- Discrete random variables are used to describe random phenomena in which only integer values can occur.
- In this section, we will learn about:
  - ☐ Bernoulli trials and Bernoulli distribution
  - ☐ Binomial distribution
  - ☐ Geometric and negative binomial distribution
  - ☐ Poisson distribution

Bernoulli Trials and Bernoulli Distribution [Discrete Dist'n]

- Bernoulli Trials:
  - Consider an experiment consisting of n trials, each can be a success or a failure.
    - Let X_j = 1 if the jth experiment is a success
    - and X_j = 0 if the jth experiment is a failure
  - The Bernoulli distribution (one trial):

$$p_j(x_j) = p(x_j) = \begin{cases} p, & x_j = 1,\ j = 1,2,...,n \\ 1-p = q, & x_j = 0, j = 1,2,...,n \\ 0, & \text{otherwise} \end{cases}$$

  - where $E(X_j) = p$ and $V(X_j) = p(1-p) = pq$
- Bernoulli process:
  - The n Bernoulli trials where trails are independent:

$$p(x_1,x_2,\ldots, x_n) = p_1(x_1)p_2(x_2) \ldots p_n(x_n)$$

Binomial Distribution [Discrete Dist'n]

- The number of successes in n Bernoulli trials, X, has a binomial distribution.

$$p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0,1,2,...,n \\ 0, & \text{otherwise} \end{cases}$$

The number of outcomes having the required number of successes and failures

Probability that there are x successes and (n-x) failures

  - The mean, $E(x) = p + p + \ldots + p = n*p$
  - The variance, $V(X) = pq + pq + \ldots + pq = n*pq$

Geometric & Negative Binomial Distribution

[Discrete Dist'n]

- Geometric distribution
  - The number of Bernoulli trials, *X*, to achieve the 1st success:

$$p(x) = \begin{cases} q^{x-1}p, & x = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

  - *E(x) = 1/p, and V(X) = q/p²*

- Negative binomial distribution
  - The number of Bernoulli trials, X, until the *k*th success
  - If Y is a negative binomial distribution with parameters p and k, then:

$$p(x) = \begin{cases} \binom{y-1}{k-1} q^{y-k}p^{k}, & y = k, k+1, k+2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

  - *E(Y) = k/p, and V(X) = kq/p²*

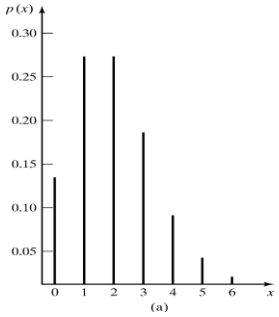
Poisson Distribution

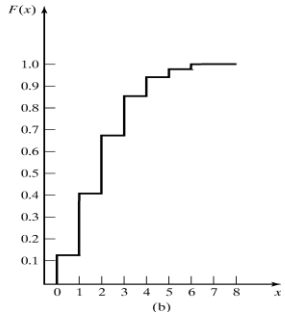
[Discrete Dist'n]

- Poisson distribution describes many random processes quite well and is mathematically quite simple.
  - where $\alpha > 0$, pdf and cdf are:

$$p(x) = \begin{cases} \dfrac{e^{-\alpha}\alpha^{x}}{x!}, & x = 0,1,\ldots \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \sum_{i=0}^{x} \frac{e^{-\alpha}\alpha^{i}}{i!}$$

  - *E(X) = α = V(X)*





Poisson Distribution [Discrete Dist'n]

- Example: A computer repair person is "beeped" each time there is a call for service. The number of beeps per hour ~ Poisson(α = 2 per hour).
  - The probability of three beeps in the next hour:

    $p(3) = e^{-2}2^3/3! = 0.18$

    also, $p(3) = F(3) - F(2) = 0.857-0.677=0.18$
    - The probability of two or more beeps in a 1-hour period:

      $p(2 \text{ or more}) = 1 - p(0) - p(1)$

      $= 1 - F(1)$

      $= 0.594$

3.4 Continuous Distributions

- Continuous random variables can be used to describe random phenomena in which the variable can take on any value in some interval.
- In this section, the distributions studied are:
  - Uniform
  - Exponential
  - Normal
  - Weibull
  - Lognormal

Uniform Distribution [Continuous Dist'n]

- A random variable X is uniformly distributed on the interval (a,b), $U(a,b)$, if its pdf and cdf are:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} 0, & x \prec a \\ \dfrac{x-a}{b-a}, & a \le x \prec b \\ 1, & x \ge b \end{cases}$$

- Properties
  - $P(x_1 < X < x_2)$ is proportional to the length of the interval [$F(x_2) - F(x_1) = (x_2-x_1)/(b-a)$]
  - $E(X) = (a+b)/2$ $V(X) = (b-a)^2/12$
- U(0,1) provides the means to generate random numbers, from which random variates can be generated.

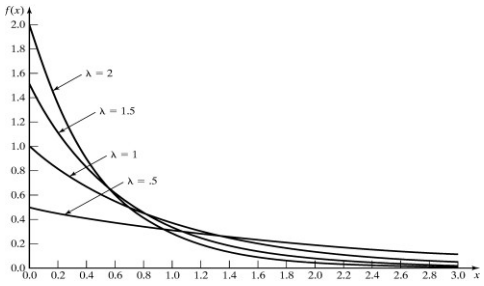
Exponential Distribution [Continuous Dist'n]

- A random variable X is exponentially distributed with

parameter $\lambda > 0$ if its pdf and cdf are:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases} \qquad F(x) = \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

- $E(X) = 1/\lambda$ $\quad V(X) = 1/\lambda^2$
- Used to model interarrival times when arrivals are completely random, and to model service times that are highly variable
- For several different exponential pdf's (see figure), the value of intercept on the vertical axis is λ, and all pdf's eventually intersect.



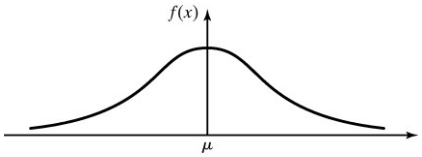
- Memoryless property
  - For all s and t greater or equal to 0:
    $P(X > s+t \mid X > s) = P(X > t)$
  - Example: A lamp ~ exp(l = 1/3 per hour), hence, on average, 1 failure per 3 hours.
    - ✓ The probability that the lamp lasts longer than its mean life is: $P(X > 3) = 1-(1-e^{-3/3}) = e^{-1} = 0.368$
    - ✓ The probability that the lamp lasts between 2 to 3 hours is:
      $P(2 <= X <= 3) = F(3) - F(2) = 0.145$
    - ✓ The probability that it lasts for another hour given it is operating for 2.5 hours:
      $P(X > 3.5 \mid X > 2.5) = P(X > 1) = e^{-1/3} = 0.717$

Normal Distribution [Continuous Dist'n]

- A random variable X is normally distributed has the pdf:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \quad -\infty < x < \infty$$

  - Mean: $-\infty < \mu < \infty$
  - Variance: $\sigma^2 > 0$
  - Denoted as $X \sim N(\mu,\sigma^2)$



- Special properties:
  - $\lim_{x\to-\infty} f(x) = 0$, and $\lim_{x\to\infty} f(x) = 0$.
  - $f(\mu-x)=f(\mu+x)$; the pdf is symmetric about μ.
  - The maximum value of the pdf occurs at $x = \mu$; the mean and mode are equal.

- **Evaluating the distribution:**
  - Use numerical methods (no closed form)
  - Independent of m and s, using the standard normal distribution:
    $Z \sim N(0,1)$

- Transformation of variables: let Z = (X - *m) / s*,

$$F(x) = P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

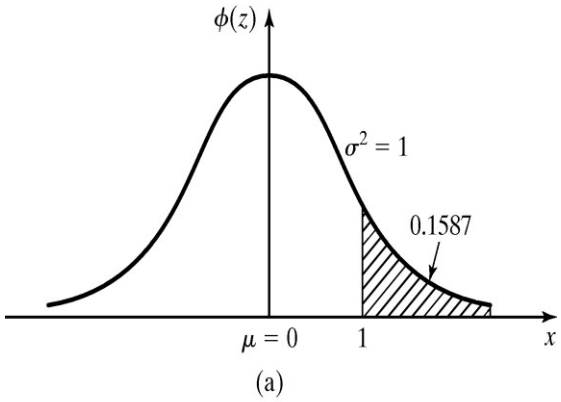
$$= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \qquad \text{, where } \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \phi(z)dz = \Phi(\frac{x-\mu}{\sigma})$$

Example: The time required to load an oceangoing vessel, *X*, is distributed as *N(12,4)*

- The probability that the vessel is loaded in less than 10 hours:

➢ Using the symmetry property, F(1) is the complement of F (-1)



(a)

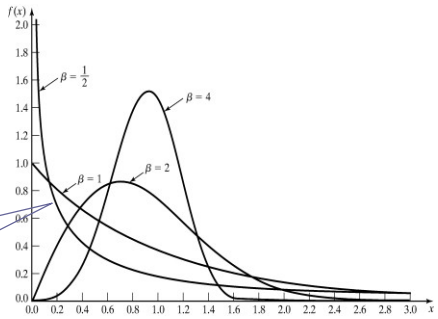
Weibull Distribution [Continuous Dist'n]

- A random variable *X* has a Weibull distribution if its pdf has the form:

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right], & x \geq \nu \\ 0, & \text{otherwise} \end{cases}$$

- 3 parameters:
  - Location parameter: υ, $(-\infty < \nu < \infty)$
  - Scale parameter: β , $(\beta > 0)$
  - Shape parameter. α, (> 0)
- Example: υ = *0* and α = *1*:

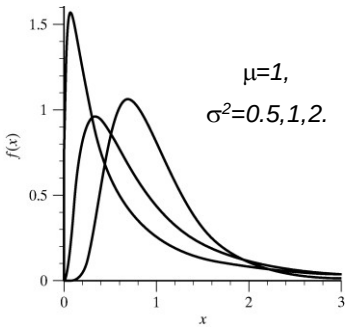
When β = *1*, *X* ~ *exp*(λ = *1*/α)



Lognormal Distribution [Continuous Dist'n]

- A random variable *X* has a lognormal distribution if its pdf has the form:

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\dfrac{(\ln x - \mu)^2}{2\sigma^2}\right], & x > 0 \\ 0, & \text{otherwise} \end{cases}$$



  - Mean E(X) = $e^{\mu+\sigma^2/2}$
  - Variance V(X) = $e^{2\mu+\sigma^2/2}(e^{\sigma^2} - 1)$

- Relationship with normal distribution
  - When $Y \sim N(\mu, \sigma^2)$, then $X = e^Y \sim$ lognormal(μ, σ^2)
  - Parameters μ and σ^2 are not the mean and variance of the lognormal

Poisson Distribution

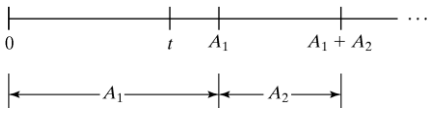
- Definition: *N(t)* is a counting function that represents the number of events occurred in *[0,t].*
- A counting process {*N(t), t>=0*} is a Poisson process with mean rate λ if:
  - Arrivals occur one at a time
  - {*N(t), t>=0*} has stationary increments
  - {*N(t), t>=0*} has independent increments
- Properties

$$P[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad \text{for } t \geq 0 \text{ and } n = 0,1,2,...$$

  - Equal mean and variance: *E[N(t)] = V[N(t)]* = λt
  - Stationary increment: The number of arrivals in time *s* to *t* is also Poisson-distributed with mean $\lambda(t-s)$

Interarrival Times [Poisson Dist'n]

- Consider the interarrival times of a Possion process (A_1, A_2, …), where A_i is the elapsed time between arrival *i* and arrival *i+1*

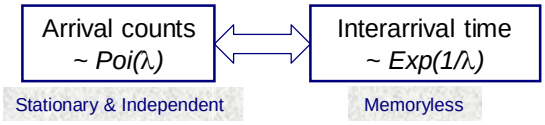


  - The 1st arrival occurs after time t iff there are no arrivals in the interval [0,t], hence:

$$P\{A_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

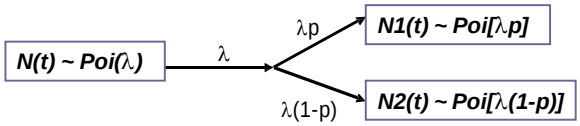
$$P\{A_1 <= t\} = 1 - e^{-\lambda t} \quad \text{[cdf of exp}(\lambda)\text{]}$$

  - Interarrival times, A_1, A_2, …, are exponentially distributed and independent with mean $1/\lambda$

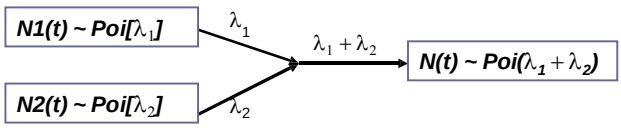


Splitting and Pooling [Poisson Dist'n]

- Splitting:
  - Suppose each event of a Poisson process can be classified as Type I, with probability *p* and Type II, with probability *1-p.*
  - *N(t) = N1(t) + N2(t)*, where *N1(t)* and *N2(t)* are both Poisson processes with rates λp and $\lambda (1-p)$



- Pooling:
  - Suppose two Poisson processes are pooled together
  - *N1(t) + N2(t) = N(t)*, where *N(t)* is a Poisson processes with rates $\lambda_1 + \lambda_2$



3.5 Poisson process;

Nonstationary Poisson Process (NSPP) [Poisson Dist'n]

- Poisson Process without the stationary increments, characterized by $\lambda(t)$, the arrival rate at time t.
- The expected number of arrivals by time t, $\Lambda(t)$:

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

- Relating stationary Poisson process $n(t)$ with rate $\lambda=1$ and NSPP $N(t)$ with rate $\lambda(t)$:
  - Let arrival times of a stationary process with rate $\lambda = 1$ be $t_1, t_2, \ldots$, and arrival times of a NSPP with rate $\lambda(t)$ be $T_1, T_2, \ldots$, we know:

$$t_i = \Lambda(T_i)$$
$$T_i = \Lambda^{-1}(t_i)$$

Nonstationary Poisson Process (NSPP) [Poisson Dist'n]

- Example: Suppose arrivals to a Post Office have rates 2 per minute from 8 am until 12 pm, and then 0.5 per minute until 4 pm.
- Let t = 0 correspond to 8 am, NSPP $N(t)$ has rate function:

$$\lambda(t) = \begin{cases} 2, & 0 \le t < 4 \\ 0.5, & 4 \le t < 8 \end{cases}$$

Expected number of arrivals by time t:

$$\Lambda(t) = \begin{cases} 2t, & 0 \le t < 4 \\ \int_0^4 2ds + \int_4^t 1.5ds = t + 6, & 4 \le t < 8 \end{cases}$$

- Hence, the probability distribution of the number of arrivals between 11 am and 2 pm.

$P[N(6) - N(3) = k]$ $= P[N(\Lambda(6)) - N(\Lambda(3)) = k]$
$= P[N(9) - N(6) = k]$
$= e^{(9-6)}(9-6)^k/k!$ $= e^3(3)^k/k!$

3.6 Empirical Distributions

A distribution whose parameters are the observed values in a sample of data.

- May be used when it is impossible or unnecessary to establish that a random variable has any particular parametric distribution.
- Advantage: no assumption beyond the observed values in the sample.
- Disadvantage: sample might not cover the entire range of possible values.

UNIT – 4 : QUEUING MODELS

Introduction

- Simulation is often used in the analysis of queueing models.
- A simple but typical queueing model:
- Queueing models provide the analyst with a powerful tool for designing and evaluating the performance of queueing systems.
- Typical measures of system performance:
  - Server utilization, length of waiting lines, and delays of customers
  - For relatively simple systems, compute mathematically
  - For realistic models of complex systems, simulation is usually required.
  -

4.1 Characteristics of Queuing Systems

Key elements of queuing systems:

Customer: refers to anything that arrives at a facility and requires service, e.g., people, machines, trucks, emails.

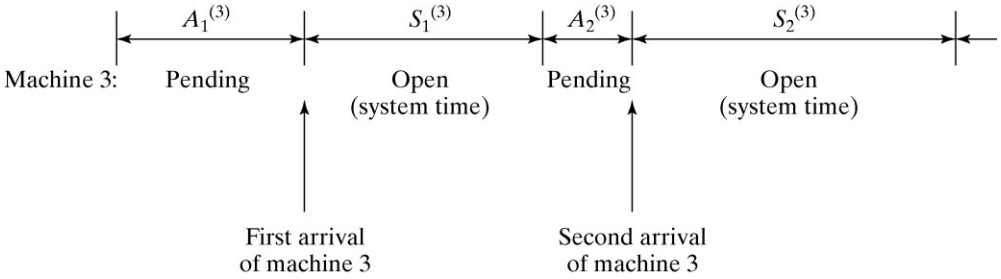
Server: refers to any resource that provides the requested service, e.g., repairpersons, retrieval machines, runways at airport.

- **Calling population**: the population of potential customers, may be assumed to be finite or infinite.
  - **Finite population model**: if arrival rate depends on the number of customers being served and waiting, e.g., model of one corporate jet, if it is being repaired, the repair arrival rate becomes zero.
  - **Infinite population model**: if arrival rate is not affected by the number of customers being served and waiting, e.g., systems with large population of potential customers.
- **System Capacity**: a limit on the number of customers that may be in the waiting line or system.
  - **Limited capacity**, e.g., an automatic car wash only has room for *10* cars to wait in line to enter the mechanism.
  - **Unlimited capacity**, e.g., concert ticket sales with no limit on the number of people allowed to wait to purchase tickets.

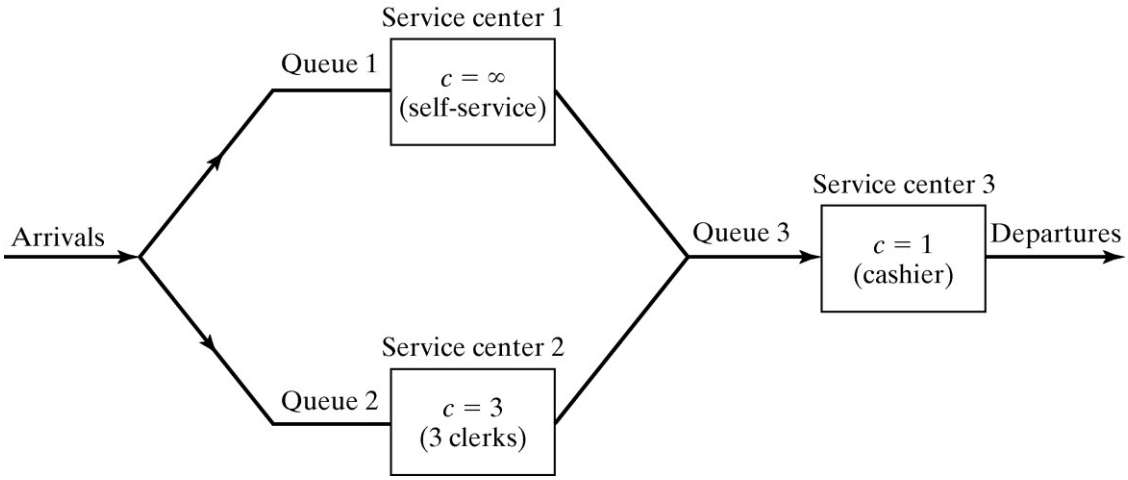
- **For infinite-population models:**
  - In terms of interarrival times of successive customers.
  - Random arrivals: interarrival times usually characterized by a probability distribution.
    - Most important model: Poisson arrival process (with rate l), where A_n represents the interarrival time between customer *n-1* and customer *n*, and is exponentially distributed (with mean *1/l*).
  - Scheduled arrivals: interarrival times can be constant or constant plus or minus a small random amount to represent early or late arrivals.
    - e.g., patients to a physician or scheduled airline flight arrivals to an airport.

- At least one customer is assumed to always be present, so the server is never idle, e.g., sufficient raw material for a machine.

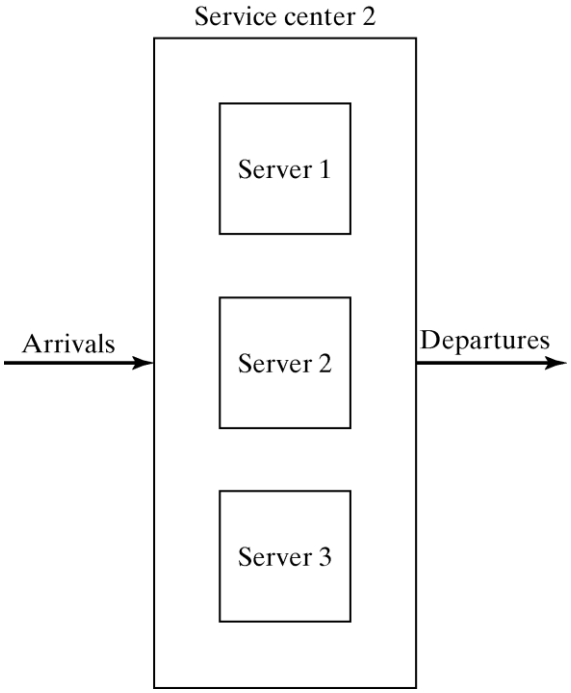
- **For finite-population models:**
  - Customer is pending when the customer is outside the queueing system, e.g., machine-repair problem: a machine is ―pending‖ when it is operating, it becomes ―not pending‖ the instant it demands service form the repairman.
  - Runtime of a customer is the length of time from departure from the queueing system until that customer‘s next arrival to the queue, e.g., machine-repair problem, machines are customers and a runtime is time to failure.
  - Let $A_1^{(i)}$, $A_2^{(i)}$, … be the successive runtimes of customer i, and $S_1^{(i)}$, $S_2^{(i)}$ be the corresponding successive system times:



- **Queue behavior:** the actions of customers while in a queue waiting for service to begin, for example:
  - Balk: leave when they see that the line is too long,
  - Renege: leave after being in the line when its moving too slowly,
  - Jockey: move from one line to a shorter line.
- **Queue discipline:** the logical ordering of customers in a queue that determines which customer is chosen for service when a server becomes free, for example:
  - First-in-first-out (FIFO)
  - Last-in-first-out (LIFO)
  - Service in random order (SIRO)
  - Shortest processing time first (SPT)
  - Service according to priority (PR).
- Service times of successive arrivals are denoted by S_1, S_2, S_3.
  - May be constant or random.
  - $\{S_1, S_2, S_3, \ldots\}$ is usually characterized as a sequence of independent and identically distributed random variables, e.g., exponential, Weibull, gamma, lognormal, and truncated normal distribution.
- A queueing system consists of a number of service centers and interconnected queues.
  - Each service center consists of some number of servers, c, working in parallel, upon getting to the head of the line, a customer takes the 1^{st} available server.
- Example: consider a discount warehouse where customers may:
  - Serve themselves before paying at the cashier:



- Wait for one of the three clerks:



- Batch service (a server serving several customers simultaneously), or customer requires several servers simultaneously.

4.2 Queueing Notation

A notation system for parallel server queues: *A/B/c/N/K*
A represents the interarrival-time distribution,
B represents the service-time distribution,
c represents the number of parallel servers,
N represents the system capacity,
K represents the size of the calling population.

- Primary performance measures of queueing systems:
  - P_n: steady-state probability of having n customers in system,
  - $P_n(t)$: probability of n customers in system at time t,

- l: arrival rate,
- l_e: effective arrival rate,
- m: service rate of one server,
- r: server utilization,
- A_n: interarrival time between customers n-1 and n,
- S_n: service time of the nth arriving customer,
- W_n: total time spent in system by the nth arriving customer,
- W_n^Q: total time spent in the waiting line by customer n,
- $L(t)$: the number of customers in system at time t,
- $L_Q(t)$: the number of customers in queue at time t,
- L: long-run time-average number of customers in system,
- L_Q: long-run time-average number of customers in queue,
- w : long-run average time spent in system per customer,
- w_Q: long-run average time spent in queue per customer.

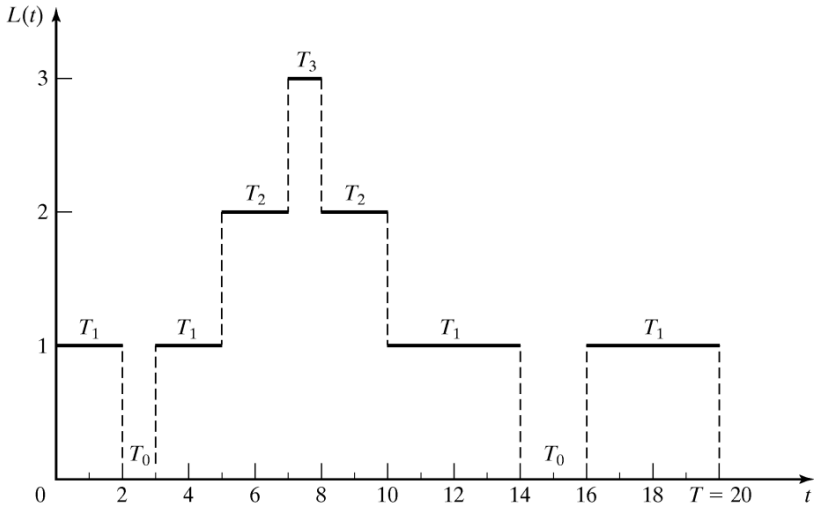
Time-Average Number in System L

- **Consider a queueing system over a period of time T,**
  - **Let T_i denote the total time during $[0,T]$ in which the system contained exactly i customers, the time-weighted-average number in a system is defined by:**
$$\hat{L} = \frac{1}{T}\sum_{i=0}^{\infty} iT_i = \sum_{i=0}^{\infty} i\left(\frac{T_i}{T}\right)$$
  - **Consider the total area under the function is $L(t)$, then,**
$$\hat{L} = \frac{1}{T}\sum_{i=0}^{\infty} iT_i = \frac{1}{T}\int_0^T L(t)dt$$
  - The long-run time-average # in system, with probability *1*:
$$\hat{L} = \frac{1}{T}\int_0^T L(t)dt \to L \quad \text{as} \quad T \to \infty$$
  - The time-weighted-average number in queue is:
$$\hat{L}_Q = \frac{1}{T}\sum_{i=0}^{\infty} iT_i^Q = \frac{1}{T}\int_0^T L_Q(t)dt \to L_Q \quad \text{as} \quad T \to \infty$$

G/G/1/N/K example: consider the results from the queueing system ($N > 4$, $K > 3$).

$$\hat{L} = [0(3) + 1(12) + 2(4) + 3(1)]/20$$
$$= 23/20 = 1.15 \text{ cusomters}$$

$$L_Q(t) = \begin{cases} 0, & \text{if } L(t) = 0 \\ L(t) - 1, & \text{if } L(t) \geq 1 \end{cases} \qquad \hat{r}_Q = \frac{0(15) + 1(4) + 2(1)}{20} = 0.3 \text{ customers}$$



Average Time Spent in System Per Customer *w*

The average time spent in system per customer, called the average system time, is:

$$\hat{w} = \frac{1}{N}\sum_{i=1}^{N} W_i$$

where $W_1, W_2, \ldots, W_N$ are the individual times that each of the N customers spend in the system during $[0,T]$.

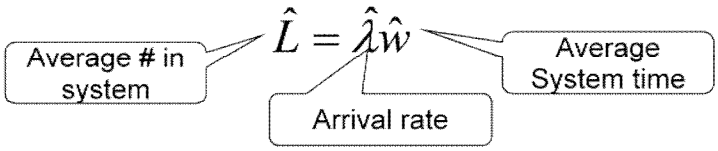
- For stable systems: $\hat{w} \to w \text{ as } N \to \infty$
- If the system under consideration is the queue alone:

$$\hat{w}_Q = \frac{1}{N}\sum_{i=1}^{N} W_i^Q \to w_Q \quad \text{as} \quad N \to \infty$$

- *G/G/1/N/K* example (cont.): the average system time is

$$\hat{w} = \frac{W_1 + W_2 + \ldots + W_5}{5} = \frac{2 + (8-3) + \ldots + (20-16)}{5} = 4.6 \text{ time units}$$

- Conservation equation (a.k.a. Little's law)



  - Holds for almost all queueing systems or subsystems (regardless of the number of servers, the queue discipline, or other special circumstances).
  - *G/G/1/N/K* example (cont.): On average, one arrival every *4* time units and each arrival spends *4.6* time units in the system. Hence, at an arbitrary point in time, there is *(1/4)(4.6) = 1.15* customers present on average.

- Definition: the proportion of time that a server is busy.
  - Observed server utilization, , is defined over a specified time interval [0,T].
  - Long-run server utilization is *r*.

For systems with long-run stability: $\hat{\rho} \to \rho \quad \text{as} \quad T \to \infty$

Server Utilization

- For *G/G/1/∞/∞* queues:
  - Any single-server queueing system with average arrival rate λ customers per time unit, where average service time $E(S) = 1/\mu$ time units, infinite queue capacity and calling population.
  - Conservation equation, $L = \lambda w$, can be applied.
  - For a stable system, the average arrival rate to the server, λ_s, must be identical to λ.
  - The average number of customers in the server is:

$$\hat{L}_s = \frac{1}{T}\int_0^T \left(L(t) - L_Q(t)\right)dt = \frac{T - T_0}{T}$$

- **In general, for a single-server queue:**

$$\hat{L}_s = \hat{\rho} \rightarrow L_s = \rho \quad \text{as} \quad T \rightarrow \infty$$

$$\text{and} \quad \rho = \lambda E(s) = \frac{\lambda}{\mu}$$

For a single-server stable queue: $\rho = \frac{\lambda}{\mu} < 1$

For an unstable queue ($l > m$), long-run server utilization is *1*.

- For *G/G/c/∞/∞* queues:
  - A system with c identical servers in parallel.
  - If an arriving customer finds more than one server idle, the customer chooses a server without favoring any particular server.
  - For systems in statistical equilibrium, the average number of busy servers, L_s, is: $L_s = \lambda E(s) = \lambda/\mu$.
  - The long-run average server utilization is:

$$\rho = \frac{L_s}{c} = \frac{\lambda}{c\mu}, \quad \text{where } \lambda < c\mu \text{ for stable systems}$$

- System performance varies widely for a given utilization ρ.
  - For example, a *D/D/1* queue where $E(A) = 1/\lambda$ and $E(S) = 1/\mu$, where:

$$L = \rho = \lambda/\mu, \quad w = E(S) = 1/\mu, \quad L_Q = W_Q = 0.$$

    - By varying λ and μ, server utilization can assume any value between *0* and *1*.
    - Yet there is never any line.
  - In general, variability of interarrival and service times causes lines to fluctuate in length.

Server Utilization and System Performance

- Example: A physician who schedules patients every *10* minutes and spends S_i minutes with the i^{th} patient:
  - Arrivals are deterministic, $A_1 = A_2 = \ldots = \lambda^{-1} = 10$.
  - Services are stochastic, $E(S_i) = 9.3$ min and $V(S_0) = 0.81$ min^2.
  - On average, the physician's utilization = $\rho = \lambda/\mu = 0.93 < 1$.
  - Consider the system is simulated with service times: $S_1 = 9$, $S_2 = 12$, $S_3 = 9$, $S_4 = 9$, $S_5 = 9$, ... The system becomes:
  - The occurrence of a relatively long service time ($S_2 = 12$) causes a waiting line to form temporarily.

Costs in Queueing Problems

- Costs can be associated with various aspects of the waiting line or servers:
  - System incurs a cost for each customer in the queue, say at a rate of *$10* per hour per customer.
    - The average cost per customer is:
    - If customers per hour arrive (on average), the average cost per hour is:
  - Server may also impose costs on the system, if a group of *c* parallel servers ($1 \le c \le \infty$) have utilization r, each server imposes a cost of *$5* per hour while busy.
    - The total server cost is: $5*c\rho$.

Steady-State Behavior of Infinite-Population Markovian Models

- Markovian models: exponential-distribution arrival process (mean arrival rate = l).
- Service times may be exponentially distributed as well (*M*) or arbitrary (*G*).
- A queueing system is in statistical equilibrium if the probability that the system is in a given state is not time dependent:

$$P(\,L(t) = n\,) = P_n(t) = P_n.$$

- Mathematical models in this chapter can be used to obtain approximate results even when the model assumptions do not strictly hold (as a rough guide).

Simulation can be used for more refined analysis (more faithful representation for complex systems).

4.4 Steady-State Behavior of Infinite-Population Markovian Models

- For the simple model studied in this chapter, the steady-state parameter, L, the time-average number of customers in the system is:

$$L = \sum_{n=0}^{\infty} nP_n$$

- Apply Little's equation to the whole system and to the queue alone:

$$w = \frac{L}{\lambda}, \quad w_Q = w - \frac{1}{\mu}$$
$$L_Q = \lambda w_Q$$

- *G/G/c/∞/∞* example: to have a statistical equilibrium, a necessary and sufficient condition is *l/(cm) < 1*.

M/G/1 Queues

- Single-server queues with Poisson arrivals & unlimited capacity.
- Suppose service times have mean *1/m* and variance s^2 and $r = l/m < 1$, the steady-state parameters of *M/G/1* queue:

$$\rho = \lambda/\mu, \quad P_0 = 1-\rho$$
$$L = \rho + \frac{\rho(1+\sigma^2\mu^2)}{2(1-\rho)}, \qquad L_Q = \frac{\rho(1+\sigma^2\mu^2)}{2(1-\rho)}$$
$$w = \frac{1}{\mu} + \frac{\lambda(1/\mu^2+\sigma^2)}{2(1-\rho)}, \quad w_Q = \frac{\lambda(1/\mu^2+\sigma^2)}{2(1-\rho)}$$

  - No simple expression for the steady-state probabilities P_0, P_1, …
  - $L - L_Q = r$ is the time-average number of customers being served.

Average length of queue, L_Q, can be rewritten as:

$$L_Q = \frac{\rho^2}{2(1-\rho)} + \frac{\lambda^2\sigma^2}{2(1-\rho)}$$

If l and m are held constant, L_Q depends on the variability, s^2, of the service times.

- Example: Two workers competing for a job, Able claims to be faster than Baker on average, but Baker claims to be more consistent,

  Poisson arrivals at rate $l = 2$ per hour ($1/30$ per minute).

  Able: $1/m = 24$ minutes and $s^2 = 20^2 = 400$ minutes^2:

$$L_Q = \frac{(1/30)^2[24^2 + 400]}{2(1-4/5)} = 2.711 \text{ customers}$$

The proportion of arrivals who find Able idle and thus experience no delay is $P_0 = 1\text{-}r = 1/5 = 20\%$.

Baker: $1/m = 25$ *minutes* and $s^2 = 2^2 = 4$ minutes^2:

$$L_Q = \frac{(1/30)^2[25^2 + 4]}{2(1-5/6)} = 2.097 \text{ customers}$$

The proportion of arrivals who find Baker idle and thus experience no delay is $P_0 = 1\text{-}r = 1/6 = 16.7\%$.

Although working faster on average, Able's greater service variability results in an average queue length about 30% greater than Baker's.

M/M/1 Queues

Suppose the service times in an *M/G/1* queue are exponentially distributed with mean $1/m$, then the variance is $s^2 = 1/m^2$.

M/M/1 queue is a useful approximate model when service times have standard deviation approximately equal to their means.

The steady-state parameters:

$$\rho = \lambda/\mu, \quad P_n = (1-\rho)\rho^n$$
$$L = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}, \quad L_Q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$
$$w = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}, \quad w_Q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)}$$

- Example: *M/M/1* queue with service rate $m=10$ customers per hour.
  - Consider how L and w increase as arrival rate, l, increases from 5 to 8.64 by increments of 20%:

If $l/m \geq 1$, waiting lines tend to continually grow in length.

Increase in average system time (w) and average number in system (L) is highly nonlinear as a function of r.

Effect of Utilization and Service Variability

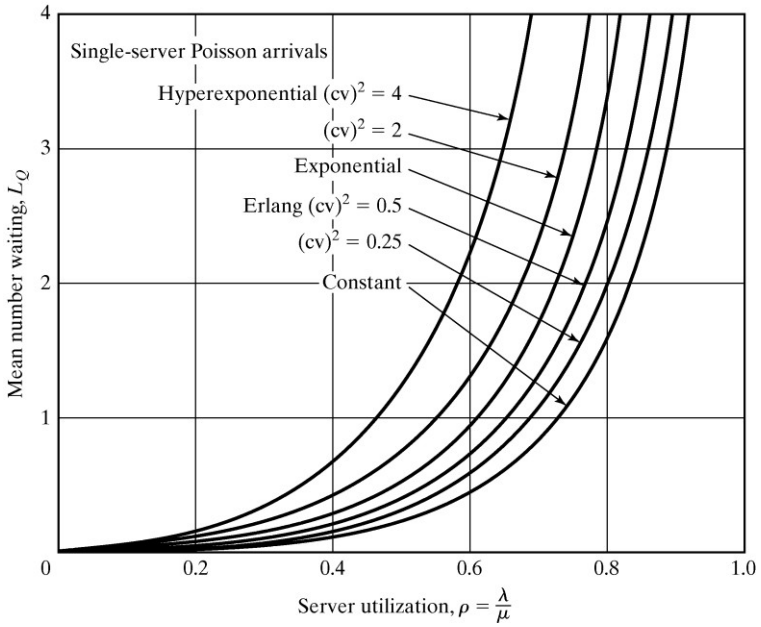
For almost all queues, if lines are too long, they can be reduced by decreasing server utilization (r) r by decreasing the service time variability (s^2).

A measure of the variability of a distribution, coefficient of variation (cv):

$$(cv)^2 = \frac{V(X)}{[E(X)]^2}$$

The larger cv is, the more variable is the distribution relative to its expected value

- Consider L_Q for any *M/G/1* queue:



Multiserver Queue

- *M/M/c/∞/∞* queue: *c* channels operating in parallel.
  - Each channel has an independent and identical exponential service-time distribution, with mean *1/m*.

- To achieve statistical equilibrium, the offered load (l/m) must satisfy $l/m < c$, where $l/(cm) = r$ is the server utilization.
- Some of the steady-state probabilities:

$$\rho = \lambda / c\mu$$

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} \right] + \left[\left(\frac{\lambda}{\mu} \right)^c \left(\frac{1}{c!} \right) \left(\frac{c\mu}{c\mu - \lambda} \right) \right] \right\}^{-1}$$

$$L = c\rho + \frac{(c\rho)^{c+1} P_0}{c(c!)(1-\rho)^2} = c\rho + \frac{\rho P(L(\infty) \geq c)}{1-\rho}$$

$$w = \frac{L}{\lambda}$$

- Other common multiserver queueing models:
  - *M/G/c/∞*: general service times and c parallel server. The parameters can be approximated from those of the *M/M/c/∞/∞* model.
  - *M/G/∞:* general service times and infinite number of servers, e.g., customer is its own system, service capacity far exceeds service demand.
  - *M/M/C/N/∞*: service times are exponentially distributed at rate m and c servers where the total system capacity is $N \geq c$ customer (when an arrival occurs and the system is full, that arrival is turned away).
- *M/M/c/∞/∞* queue: *c* channels operating in parallel.
  - Each channel has an independent and identical exponential service-time distribution, with mean *1/m*.
  - To achieve statistical equilibrium, the offered load (l/m) must satisfy $l/m < c$, where $l/(cm) = r$ is the server utilization.

Some of the steady-state probabilities:

$$\rho = \lambda / c\mu$$

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} \right] + \left[\left(\frac{\lambda}{\mu} \right)^c \left(\frac{1}{c!} \right) \left(\frac{c\mu}{c\mu - \lambda} \right) \right] \right\}^{-1}$$

$$L = c\rho + \frac{(c\rho)^{c+1} P_0}{c(c!)(1-\rho)^2} = c\rho + \frac{\rho P(L(\infty) \geq c)}{1-\rho}$$

$$w = \frac{L}{\lambda}$$

- Other common multiserver queueing models:
  - *M/G/c/∞*: general service times and c parallel server. The parameters can be approximated from those of the *M/M/c/∞/∞* model.
  - *M/G/∞:* general service times and infinite number of servers, e.g., customer is its own system, service capacity far exceeds service demand.
  - *M/M/C/N/∞*: service times are exponentially distributed at rate m and c servers where the total system capacity is *N ³ c* customer (when an arrival occurs and the system is full, that arrival is turned away).

Steady-State Behavior of Finite-Population Models

- When the calling population is small, the presence of one or more customers in the system has a strong effect on the distribution of future arrivals.
- Consider a finite-calling population model with *K* customers (*M/M/c/K/K*):
  - The time between the end of one service visit and the next call for service is exponentially distributed, (mean = *1/l*).
  - Service times are also exponentially distributed.
  - *c* parallel servers and system capacity is *K*.

Steady-State Behavior of Finite-Population Models

- Some of the steady-state probabilities:

$$P_0 = \left\{ \sum_{n=0}^{c-1} \binom{K}{n} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{K} \frac{K!}{(K-n)!c!c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n \right\}^{-1}$$

$$P_n = \begin{cases} \binom{K}{n} \left(\frac{\lambda}{\mu}\right)^n P_0, & n = 0,1,\ldots,c-1 \\ \frac{K!}{(K-n)!c!c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n, & n = c, c+1,\ldots K \end{cases}$$

$$L = \sum_{n=0} nP_n, \quad w = L/\lambda_e, \quad \rho = \lambda_e / c\mu$$

where λ_e is the long run effective arrival rate of customers to queue (or entering/exiting service)

$$\lambda_e = \sum_{n=0}^{K} (K-n)\lambda P_n$$

Steady-State Behavior of Finite-Population Models

- Example: two workers who are responsible for *10* milling machines.
  - Machines run on the average for *20* minutes, then require an average *5*-minute service period, both times exponentially distributed: *l = 1/20 and m = 1/5*.
  - All of the performance measures depend on P_0:

$$P_0 = \left\{ \sum_{n=0}^{2-1} \left(\frac{10}{n} \right) \left(\frac{5}{20} \right)^n + \sum_{n=2}^{10} \frac{10!}{(10-n)!2!2^{n-2}} \left(\frac{5}{20} \right)^n \right\}^{-1} = 0.065$$

Then, we can obtain the other P_n.

Expected number of machines in system:

$$L = \sum_{n=0}^{10} nP_n = 3.17 \text{ machines}$$

The average number of running machines:

$$K - L = 10 - 3.17 = 6.83 \text{ machines}$$

4.5 Steady state behavior of M/G/1 queue; Networks of queues

Networks of Queues

- Many systems are naturally modeled as networks of single queues: customers departing from one queue may be routed to another.
- The following results assume a stable system with infinite calling population and no limit on system capacity:
  - Provided that no customers are created or destroyed in the queue, then the departure rate out of a queue is the same as the arrival rate into the queue (over the long run).
  - If customers arrive to queue i at rate l_i, and a fraction $0 \leq p_{ij} \leq 1$ of them are routed to queue j upon departure, then the arrival rate form queue i to queue j is $l_i p_{ij}$ (over the long run).
  - The overall arrival rate into queue j:

$$\lambda_j = a_j + \sum_{\text{all } i} \lambda_i p_{ij}$$

Arrival rate from outside the network

Sum of arrival rates from other queues in network

  - If queue j has $c_j < \infty$ parallel servers, each working at rate m_j, then the long-run utilization of each server is $r_j = l_j/(cm_j)$ (where $r_j < 1$ for stable queue).
  - If arrivals from outside the network form a Poisson process with rate a_j for each queue j, and if there are c_j identical servers delivering exponentially distributed service times with mean $1/m_j$, then, in steady state, queue j behaves likes an $M/M/c_j$ queue with arrival rate

$$\lambda_j = a_j + \sum_{\text{all } i} \lambda_i p_{ij}$$

Discount store example:

- Suppose customers arrive at the rate *80* per hour and *40%* choose self-service. Hence:
  - Arrival rate to service center 1 is $l_1 = 80(0.4) = 32$ per hour
  - Arrival rate to service center 2 is $l_2 = 80(0.6) = 48$ per hour.
- $c_2 = 3$ clerks and $m_2 = 20$ customers per hour.
- The long-run utilization of the clerks is:

$$r_2 = 48/(3*20) = 0.8$$

- All customers must see the cashier at service center 3, the overall rate to service center 3 is $l_3 = l_1 + l_2 = 80$ per hour.
  - If $m_3 = 90$ per hour, then the utilization of the cashier is:

$$r_3 = 80/90 = 0.89$$

UNIT – 5 : RANDOM-NUMBER GENERATION, RANDOM-VARIATE GENERATION

RANDOM-NUMBER GENERATION

Random numbers are a necessary basic ingredient in the simulation of almost all discrete systems. Most computer languages have a subroutine, object, or function that will generate a random number. Similarly simulation languages generate random numbers that arc used to generate event limes and other random variables.

5.1 Properties of Random Numbers

A sequence of random numbers, W1, W2, .. , must have two important statistical properties, uniformity and independence. Each random number *Ri,* is an independent sample drawn from a continuous uniform distribution between zero and 1. That is, the pdf is given by

$$F(x)=\begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

This density function is shown in Figure 7.1. The expected value of each *Ri,* is:

$$E(R) = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1$$

and the variance is given by

$$V(R)=\int_0^1 x^2 dx - [E(R)]^2 = \frac{x^3}{3}\Big|_0^1 - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

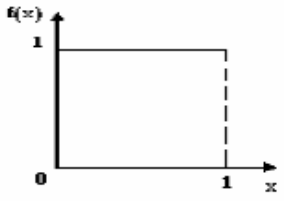


Figure 7.1* The: pdf for random numbers.

Some consequences of the uniformity and independence properties are the following:

1. If the interval (0,1) is divided into *n* classes, or subintervals of equal length, the expected number of observations m each interval ii *N/n* where A' is the total number of observations.

2. The probability of observing a value in a particular interval is of the previous values drawn

5.2**Generation of Pseudo-Random Numbers**

Pseudo means false,so false random numbers are being generated.The goal of any generation scheme, is to produce a sequence of numbers belween zero and 1 which simulates, or initates, the ideal properties of uniform distribution and independence as closely as possible.

When generating pseudo-random numbers, certain problems or errors can occur. These errors, or departures from ideal randomness, are all related to the properties stated previously.

Some examples include the following

1. The generated numbers may not be uniformly distributed.

2. The generated numbers may be discrete -valued instead continuous valued

3. The mean of the generated numbers may be too high or too low.

4. The variance of the generated numbers may be too high or low

5. There may be dependence. The following are examples:

(a) Autocorrelation between numbers.

(b) Numbers successively higher or lower than adjacent numbers.

(c) Several numbers above the mean followed by several numbers below the mean.

Usually, random numbers are generated by a digital computer as part of the simulation.

Numerous methods can be used to generate the values. In selecting among these methods, or routines, there are a number of important considerations.

1. The routine should be fast. . The total cost can be managed by selecting a computationally efficient method of random-number generation.
2. The routine should be portable to different computers, and ideally to different programming languages .This is desirable so that the simulation program produces the same results wherever it is executed.
3. The routine should have a sufficiently long cycle. The cycle length, or period, represents the length of the random-number sequence before previous numbers begin to repeat themselves in an earlier order. Thus, if 10,000 events are to be generated, the period should be many times that long,

   A special case cycling is degenerating. A routine degenerates when the same random numbers appear repeatedly. Such an occurrence is certainly unacceptable. This can happen rapidly with some methods.

4. The random numbers should be replicable. Given the starting point (or conditions), it should be possible to generate the same set of random numbers, completely independent of the system that is being simulated. This is helpful for debugging purpose and is a means of facilitating comparisons between systems.
5. Most important, and as indicated previously, the generated random numbers should closely approximate the ideal statistical properties of uniformity and independences

5.3**Techniques for Generating Random Numbers**

The linear congruential method, initially proposed by Lehmer [1951], produces a sequence of integers, *X*\, *X2*,... between zero and *m* — 1 according to the following recursive relationship:

$$X_{i+1} = (a X_i + c) \bmod m, \quad i = 0,1,2,... \quad (7.1)$$

The initial value X0 is called the seed, *a* is called the constant multiplier, c is the increment, and

m is the modulus.

If $c \neq 0$ in Equation (7.1), the form is called the ***mixed congruential method***. When $c = 0$, the form is known as the ***multiplicative congruential method***. The selection of the values for a, *c*, m and Xo drastically affects the statistical properties and the cycle length. . An example will illustrate how this technique operates.

EXAMPLE 4.1

Use the linear congruential method to generate a sequence of random numbers with *X0* = 27, *a*= 17, *c* = 43, and *m* = 100. Here, the integer values generated will all be between zero and 99 because of the value of the modulus . These random integers should appear to be uniformly distributed the integers zero to 99.Random numbers between zero and 1 can be generated by

$R_i = X_i/m, \quad i = 1,2,\ldots$ **(7.2)**

The sequence of Xi and subsequent Ri values is computed as follows:

X0 = 27

X1 = (17.27 + 43) mod 100 = 502 mod 100 = 2

R1=2⁄100=0. 02

X2 = (17 • 2 + 43) mod 100 = 77 mod 100 = 77

R2=77⁄100=0. 77

*[3pt] X3 = (17•77+ 43) mod 100 = 1352 mod 100 = 52

R3=52⁄100=0. 52

First, notice that the numbers generated from Equation (7.2) can only assume values from the set I = {0,1 /m, 2/m,..., (m — l)/m), since each *Xi* is an integer in the set {0,1,2,..., m — 1}. Thus, each *Ri* is discrete on I, instead of continuous on the interval [0, 1], This approximation appears to be of little consequence, provided that the modulus m is a very large integer. (Values such as $m = 231 — 1$ and $m = 248$ are in common use in generators appearing in many simulation languages.) By maximum density is meant that the values assumed by *Ri* = 1, 2,..., leave no large gaps on [0,1]

Second, to help achieve maximum density, and to avoid cycling (i.e., recurrence of the same sequence of generated numbers) in practical applications, the generator should have the largest possible period. Maximal period can be achieved by the proper choice of *a*, c, m, and *X0* .

- For m a power of 2, say $m = 2b$ and $c \neq 0$, the longest possible period is $P = m = 2b$, which is achieved provided that *c* is relatively prime to m (that is, the greatest common factor of c and m i s l), and =a = l+4k, where *k* is an integer.

- For m a power of 2, say $m = 2b$ and $c = 0$, the longest possible period is $P = m⁄4 = 2b-2$, which is achieved provided that the seed X0 is odd and the multiplier ,a, is given by a=3+8K , for some

  K=0,1,..

- For m a prime number and c=0, the longest possible period is P=m-1, which is achieved provided that the multiplier , a, has the property that the smallest integer k such that

$a^k - 1$is divisible by m is k= m-1.

EXAMPLE 4.3

Let $m = 102 = 100$, $a = 19$, $c = 0$, and $X0 = 63$, and generate a sequence c random integers using Equation (7.1).

$X0 = 63$

$X1 = (19)(63) \bmod 100 = 1197 \bmod 100 = 97$

$X2 = (19)(97) \bmod 100 = 1843 \bmod 100 = 43$

$X3 = (19)(43) \bmod 100 = 817 \bmod 100 = 17$

.

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When m is a power of 10, say $m = 10b$, the modulo operation is accomplished by saving the *b* rightmost (decimal) digits.

EXAMPLE 4.4

Let a = 75 = 16,807, $m = 231-1 = 2,147,483,647$ (a prime number), and c= 0. These choices satisfy the conditions that insure a period of $P = m— 1$. Further, specify a seed, $XQ = 123,457$.

The first few numbers generated are as follows:

$X^1 = 7^5(123,457) \bmod (2^{31} - 1) = 2,074,941,799 \bmod (2^{31} - 1)$

$X^1 = 2,074,941,799$

$R^1 = X^1/2^{31}$

$X_2 = 7^5(2,074,941,799) \bmod (2^{31} - 1) = 559,872,160$

$R_2 = X^2/2^{31} = 0.2607$

$X_3 = 7^5(559,872,160) \bmod (231 - 1) = 1,645,535,613$

$R_3 = X^3/2^{31} = 0.7662$

Notice that this routine divides by $m + 1$ instead of m ; however, for sucha large value of m , the effect is negligible.

Combined Linear Congruential Generators

As computing power has increased, the complexity of the systems that we are able to simulate has also increased.

One fruitful approach is to combine two or more multiplicative congruen-tial generators in such a way that the combined generator has good statistical properties and a longer period. The following result from L'Ecuyer [1988] suggests how this can be done:

If Wi, 1 , Wi , 2. . . , *W i,k* are any independent, discrete-valued random variables (not necessarily identically distributed), but one of them, say Wi, 1, is uniformly distributed on the integers 0 to mi — 2, then

$$Wi = \left(\sum_{j=1}^{k} W_{i,j} \right) \bmod m_1 - 1$$

is uniformly distributed on the integers 0 to mi — 2.

To see how this result can be used to form combined generators, let Xi,1, X i,2,..., X i,k be the i th output from *k* different multiplicative congru-ential generators, where the *j* th generator has prime modulus *mj*, and the multiplier *aj* is chosen so that the period is m*j* — 1. Then the j'th generator is producing integers *Xi,j* that are approximately uniformly distributed on 1 to *mj* - 1, and *Wi,j = X i,j* — 1 is approximately uniformly distributed on 0 to m*j* - *2.* L'Ecuyer [1988] therefore suggests combined generators of the form

$$\sum_{j=1}^{k} (-1)^{j-1} X_{i\,j} \quad \bmod \; m1 - 1$$

ith

$$R\,i = \begin{cases} Xi/m1 , & Xi > 0 \\ m1 - 1 / & m1\ Xi = 0 \end{cases}$$

Notice that the " (-1)j-1 "coefficient implicitly performs the subtraction X i,1-1; for example,

if k = 2, then $(-1)^{\circ}(X_{i1} - 1) - (-1)^{1}(X_{i2} - 1) = \Sigma^{2}_{j=1}(-1)^{j-1} X_{i,j}$

The maximum possible period for such a generator is

$$\frac{(m1 - 1)(m2 - 1) - - - (mk - 1)}{2^{k-1}}$$

which is achieved by the following generator:

EXAMPLE 4.5

For 32-bit computers, L'Ecuyer [1988] suggests combining $k = 2$ generators with $m_1 = 2147483563$, $a_1 = 40014$, $m_2 = 2147483399$, and $a_2 = 40692$. This leads to the following algorithm:

1. Select seed $X_{1,0}$ in the range [1, 2147483562] for the first generator, and seed $X_{2.o}$ in the range [1, 2147483398].

   Set j =0.

2. Evaluate each individual generator.

   $X_{1, j+1} = 40014X_{1,j} \bmod 2147483563$

   $X_{2,j+i} = 40692X_{2,j} \bmod 2147483399$

3 Set

   $Xj+1 = (X\ 1, j+1 - X\ 2\ j+1) \bmod 2147483562$

4. Return

   $R\ j+1 = Xj+1 / 2147483563 , X\ j+1 > 0$

   2147483563 / 2147483563. X j +1= 0

5. Set $j = j + 1$ and go to step 2.

5.4Tests for Random Numbers

The desirable properties of random numbers — uniformity and independence To insure that these desirable properties are achieved, a number of tests can be performed

(fortunately, the appropriate tests have already been conducted for most commercial simulation software}. The tests can be placed in two categories according to the properties of interest,

The first entry in the list below concerns testing for uniformity. The second through fifth entries concern testing for independence. The five types of tests

1. **Frequency test** Uses the Kolmogorov-Smirnov or the chi- square test to compare the distribution of the set of numbers generated to a uniform distribution.
2. **Runs test**. Tests the runs up and down or the runs above, and below the mean by comparing the actual values to expected values. The statistic for comparison is the chi-square.
3. **Autocorrelation test** Tests the correlation between numbers and compares the sample correlation to the expected correlation of zero.
4. **Gap test**. Counts the number of digits that appear between repetitions of particular digit and then uses the Kolmogorov-Smirnov test to compare with the expected size of gaps,
5. **Poker test** . Treats numbers grouped together as a poker hand. Then the hands obtained are compared to what is expected using the chi-square test.

In testing for uniformity, the hypotheses are as follows:

H0: Ri ~ U/[0,1]

H1: Ri ~U/[0,l]

The null hypothesis, H0 reads that the numbers are distributed uniformly on the interval [0,1]. Failure to reject the null hypothesis means that no evidence of nonuniformity has been detected on the basis of this test. This does not imply that further testing of the generator for uniformity is unnecessary.

In testing for independence, the hypotheses are as follows:

H0: Ri ~ independently

H1:Ri~independently

This null hypothesis H0 reads that the numbers are independent. Failure to reject the null hypothesis means that no evidence of dependence has been detected on the basis of this test. This does not imply that further testing of the generator for independence is unnecessary.

For each test, a level of significance *a* must be stated. The level *a* is the probability of rejecting the null hypothesis given that the null hypothesis is true, or

a = *P* (reject H0 |H0 true)

The decision maker sets the value of & for any test. Frequently, *a* is set to 0.01 orO.05. If several tests are conducted on the same set of numbers, the probability of rejecting the null hypothesis on at least one test, by chance alone [i.e., making a Type I (a) error], increases. Say that *a*= 0.05 and that five different tests are conducted on a sequence of numbers. The probability of rejecting the null hypothesis on at least one test, by chance alone, may be as large as 0.25.

Frequency Tests

A basic test that should always be performed to validate a new generator is the test of uniformity. Two different methods of testing are available. They are the **Kolmogorov-Smirnov** and the **chi-square test**. Both of these tests measure the degree of agreement between the distribution of a sample of generated random numbers and the theoretical uniform distribution. Both tests are on the null hypothesis of no significant difference between the sample distribution and the theoretical distribution.

1. **The Kolmogorov-Smirnov test**. This test compares the continuous cdf, F(X), of the uniform distribution to the empirical cdf, SN(x), of the sample of N observations. By definition,

$$F(x) = x, \quad 0 <= x <= 1$$

If the sample from the random-number generator is R1 R2, ,• • •, RN, then the empirical cdf, SN(X), is defined by

$$SN(X) = \frac{\text{number of } R1\ R2, ,\bullet\bullet\bullet, Rn \text{ which are } <= x}{N}$$

As N becomes larger, SN(X) should become a better approximation to F(X) , provided that the null hypothesis is true.

The **Kolmogorov-Smirnov test** is based on the largest absolute deviation between F(x) and SN(X) over the range of the random variable. That is.it is based on the statistic

$$D = \max | F(x) - SN(x)| \quad (7.3)$$

For testing against a uniform cdf, the test procedure follows these steps:

Step 1. Rank the data from smallest to largest. Let R(i) denote the i th smallest observation, so that

$$R(1) <= R(2) <= \bullet\bullet\bullet <= R(N)$$

Step 2. Compute

$$D^{+} = \max_{1<=i<=N} \{ i/N - R(i) \}$$

$$D^- = \max_{1 <= i <= N} \{ i/N - R_{(i)} \}$$

Step3. Compute D = max(D+, D-).

Step 4. Determine the critical value, Da, from Table A.8 for the specified significance level a and the given sample size N.

Step 5. If the sample statistic D is greater than the critical value, Da, the null hypothesis that the data are a sample from a uniform distribution is rejected.

If D <= Da, conclude that no difference has been detected between the true distribution of **{** R1 R2, ,• • •, Rn **}** and the uniform distribution.

EXAMPLE 4.6

Suppose that the five numbers 0.44,0.81,0.14,0.05,0.93 were generated, and it is desired to perform a test for uniformity using the Kolmogorov-Smirnov test with a level of significance a of 0.05.

First, the numbers must be ranked from smallest to largest. The calculations can be facilitated by use of Table 7.2. The top row lists the numbers from smallest (R(1)) to largest (R(n)) .The computations for D+, namely i /N -R(i} and for D-, namely R(i) - (i - l) / N, are easily accomplished using Table 7.2. The statistics are computed as D+ = 0.26 and D- = 0.21. Therefore, D = max{0.26, 0.21} = 0.26. The critical value of **D,** obtained from Table A.8 for a = 0.05 and N = 5, is 0.565. Since the computed value, 0.26, is less than the tabulated critical value, 0.565, the hypothesis of no difference between the distribution of the generated numbers and the uniform distribution is not rejected.

Table 7.2. Calculations for Kolmogorov-Smirnov Test

$R_{(i)}$	0.05	0.14	0.44	0.81	0.93
i/N	0.20	0.40	0.60	0.80	1.00
$i/N - R_{(i)}$	0.15	0.26	0.16	—	0.07
$R_{(i)} - (i-1)/N$	V 0.05	—	0.04	0.21	0.13

The calculations in Table 7.2 are illustrated in Figure 7.2, where the empirical cdf, SN(X), is compared to the uniform cdf, F(x). It can be seen that D^+ is the largest deviation of SN(x) above F(x), and that D^- is the largest deviation of SN(X) below F(x). For example, at R(3) the value of D^+ is given by 3/5 - R(3) = 0.60 - 0.44 =0.16 and of D^- is given by R(3) = 2/5 = 0.44 - 0.40 = 0.04. Although the test statistic D is defined by Equation (7.3) as the maximum deviation over all x, it can be seen from Figure 7.2 that the maximum deviation will always occur at one of the jump points R(1) , R(2) . . . , and thus the deviation at other values of x need not be considered.

2. The chi-square test. The chi-square test uses the sample statistic

$$X_0^2 = \sum_{n} (Oi - Ei)^2/Ei$$

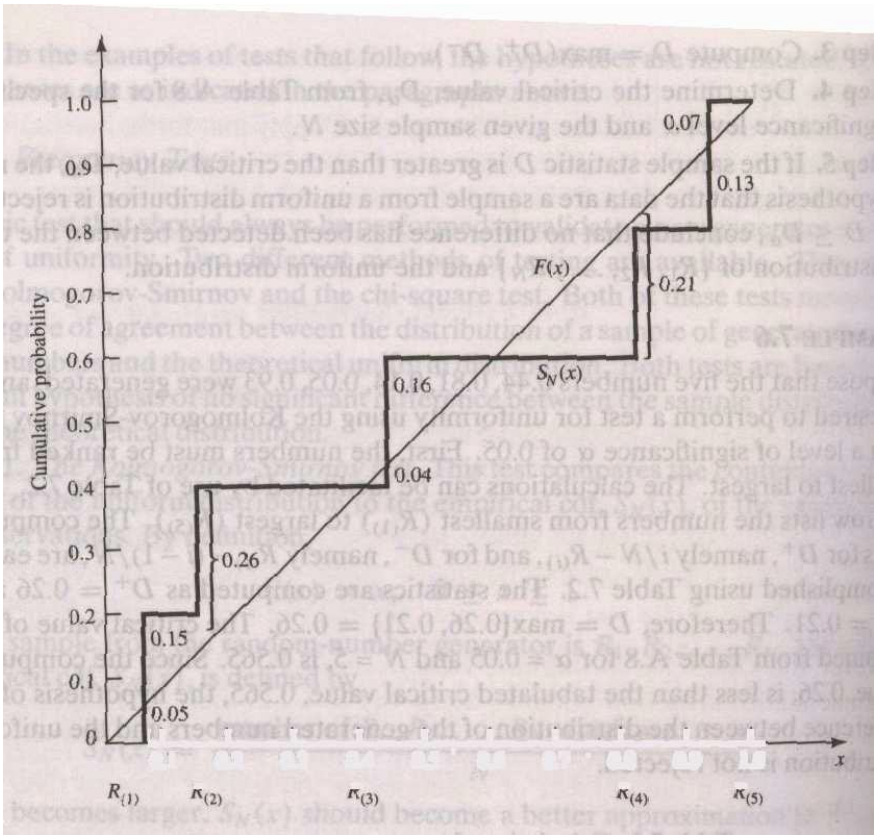


Figure 7.2. Comparison of F(x) and SN(X)

where Oi; is the observed number in the *i* th class, *Ei* is the expected number in the ith class, and *n* is the number of classes. For the uniform distribution, *Ei* the expected number in each class is

given by $Ei = N/n$ for equally spaced classes, where *N* is the total number of observations.

It can be shown that the sampling distribution of X_0^2 is approximately the chi-square

distribution with *n* - 1 degrees of freedom

EXAMPLE 4.7

Use the chi-square test with $a = 0.05$ to test whether the data shown below are uniformly distributed. Table 7.3 contains the essential computations. The test uses n = 10 intervals of equal length, namely

[0, 0.1), [0.1, 0.2), . . . , [0.9, 1.0). The value of X_0^2 is 3.4. This is compared with the critical value $X^2_{0.05,9} = 16.9$.Since X_0^2 is much smaller than the tabulated value of $X^2_{0.05,9}$, the null hypothesis of a uniform distribution is not rejected.

0.34	0.90	0.25	0.89	0.87	0.44	0.12	0.21	0.46	0.67
0.83	0.76	0.79	0.64	0.70	0.81	0.94	0.74	0.22	0.74
0.96	0.99	0.77	0.67	0.56	0.41	0.52	0.73	0.99	0.02
0.47	0.30	0.17	0.82	0.56	0.05	0.45	0.31	0.78	0.05
0.79	0.71	0.23	0.19	0.82	0.93	0.65	0.37	0.39	0.42
0.99	0.17	0.99	0.46	0.05	0.66	0.10	0.42	0.18	0.49
0.37	0.51	0.54	0.01	0.81	0.28	0.69	0.34	0.75	0.49
0.72	0.43	0.56	0.97	0.30	0.94	0.96	0.58	0.73	0.05
0.06	0.39	0.84	0.24	0.40	0.64	0.40	0.19	0.79	0.62
0.18	0.26	0.97	0.88	0.64	0.47	0.60	0.11	0.29	0.78

Both the Kolmogorov-Smirnov and the chi-square test are acceptable for testing the uniformity of a sample of data, provided that the sample size is large. However, the Kolmogorov-Smirnov test is the more powerful of the two and is recommended. Furthermore, the Kolmogorov-Smirnov test can be applied to small sample sizes, whereas the chi-square is valid only for large samples, say N>=50.

Imagine a set of 100 numbers which are being tested for independence where the first 10 values are in the range 0.01-0.10, the second 10 values are in the range 0.11-0.20, and so on. This set of numbers would pass the frequency tests with ease, but the ordering of the numbers produced by the generator would not be random. The tests in the remainder of this chapter are concerned with the independence of random numbers which are generated. The presentation of the tests is similar to that by Schmidt and Taylor [1970].

Table 7.3. Computations for Chi-Square Test

Interval	O_i	E_i	$O_i - E_i$	$(O_i -$	$(O_i - E_i)^2/E_i^2$
1	8	10	-2	4	0.4
2	8	10	—2	4	0.4
3	10	10	0	0	0.0
4	9	10	-1	1	0.1
5	12	10	2	4	0.4
6	8	10	-2	4	0.4
7	10	10	0	0	0.0
8	14	10	4	16	1.6
9	10	10	0	0	0.0
10	11	10	1	1	0.1
	100	100	0		34

Runs Tests

1. Runs up and runs down. Consider a generator that provided a set of 40 numbers in the following sequence:

0.08 0.09 0.23 0.29 0.42 0.55 0.58 0.72 0.89 0.91

0.11 0.16 0.18 0.31 0.41 0.53 0.71 0.73 0.74 0.84

0.02 0.09 0.30 0.32 0.45 0.47 0.69 0.74 0.91 0.95

0.12 0.13 0.29 0.36 0.38 0.54 0.68 0.86 0.88 0.91

Both the Kolmogorov-Smirnov test and the chi-square test would indicate that the numbers are uniformly distributed. However, a glance at the ordering shows that the numbers are successively larger in blocks of 10 values. If these numbers are rearranged as follows, there is far less reason to doubt their independence

0.41 0.68 0.89 0.84 0.74 0.91 0.55 0.71 0.36 0.30

0.09 0.72 0.86 0.08 0.54 0.02 0.11 0.29 0.16 0.18

0.88 0.91 0.95 0.69 0.09 0.38 0.23 0.32 0.91 0.53

0.31 0.42 0.73 0.12 0.74 0.45 0.13 0.47 0.58 0.29

The runs test examines the arrangement of numbers in a sequence to test the hypothesis of

independence.

Before defining a run, a look at a sequence of coin tosses will help with some terminology. Consider the following sequence generated by tossing a coin 10 times:

H T T H H T T T H T

There are three mutually exclusive outcomes, or events, with respect to the sequence. Two of the possibilities are rather obvious. That is, the toss can result in a head or a tail. The third possibility is "no event." The first head is preceded by no event and the last tail is succeeded by no event. Every sequence begins and ends with no event.

A run is defined as a succession of similar events preceded and followed by a different event. The length of the run is the number of events that occur in the run. In the coin-flipping example above there are six runs. The first run is of length one, the second and third of length two, the fourth of length three. and the fifth and sixth of length one.

There are two possible concerns in a runs test for a sequence of number The number of runs is the first concern and the length of runs is a second concern. The types of runs counted in the first case might be runs up and runs down. An up run is a sequence of numbers each of which is succeeded by a larger number. Similarly, a down run is a sequence of numbers each of which is succeeded by a smaller number. To illustrate the concept, consider the following sequence of 15 numbers:

-0.87 +0.15 +0.23 +0.45 -0.69 -0.32 -0.30 +0.19 -.24 +0.18 +0.65 +0.82 -0.93

+0.22 0.81

The numbers are given a " + " or a " — " depending on whether they are followed by a larger number or a smaller number. Since there are 15 numbers, and they are all different, there will be 14 +'s and — 's. The last number is followed by "no event" and hence will get neither a + nor a — . The sequence of 14 +s and — 's is as follows:

- + + + - - - + - + + - +

Each succession of + 's and — 's forms a run. There are eight runs. The first run is of length one. the second and third are of length three, and so on. Further, there are four runs up and four runs down.

There can be too few runs or too many runs. Consider the following sequence of numbers:

0.08 0.18 0.23 0.36 0.42 0.55 0.63 0.72 0.89 0.91

This sequence has one run, a run up. It is unlikely that a valid random-number generator would produce such a sequence. Next, consider the following sequence

0,08 0.93 0.15 0.96 0.26 0.84 0.28 0.79 0.36 0.57

This sequence has nine runs, five up and four down. It is unlikely that a sequence of lO numbers would have this many runs. What is more likely is that the number of runs will be somewhere between the two extremes. These two extremes can be formalized as follows: if N is the number of numbers in a sequence, the maximum number of runs is N — I and the minimum number of runs is one.
If a is the total number of runs in a truly random sequence, the mean and variance of a are given by

$\mu_a = 2N - 1 / 3$ (7.4)

and

$\sigma_a^2 = 16N - 29 / 90$ (7.5)

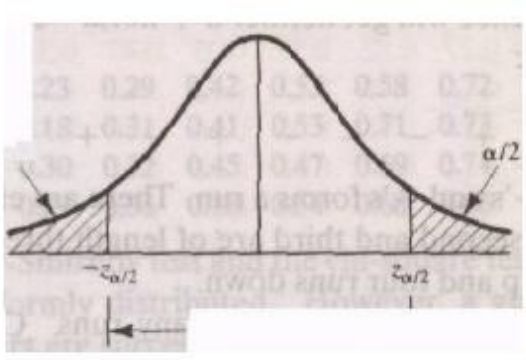
For $N > 20$, the distribution of a is reasonably approximated by a normal distribution, $N(\mu_a , \sigma_a^2)$ This approximation can be used to test the independence of numbers from a generator. In that case the standardized normal test statistic is developed by subtracting the mean from the observed number of runs, a , and dividing by the standard deviation. That is, the test statistic is

$Z_0 = a - \mu_a / \sigma_a$

Substituting Equation (7,4) for μ_a and the square root of Equation (7.5) for σ_a yields

$$Z_0 = a - [(2N - 1)/3] / sqrt((16N – 29) / 90)$$

where Zo ~ *N(0* 1). Failure to reject the hypothesis of independence occur when — *za / 2* <= Zo <= *za / 2* where *a* is the level of significance. The critical values and rejection region are shown in Figure 7.3.



Failure to reject Figure 7.3.

2. Runs above and below the mean. The test for runs up and runs down is not completely adequate to assess the independence of a group of numbers.

Consider the following 40 numbers:

0.63 0.72 0.79 0.81 0.52 0.94 0.8.1 0.93 0.87 0.67

0.54 0.83 0.89 0.55 0.88 0.77 0.74 0.95 0.82 0.86

0.43 0.32 0.36 0.18 0.08 0.19 0.18 0.27 0.36 0.34

0.31 0.45 0.49 0.43 0.46 0.35 0.25 0.39 0.47 0.41

The sequence of runs up and runs down is as follows:

+ + + - + - + - - - + + - + - - + - + - - + - - + - + + - - + + - + - - + + -

This sequence is exactly the same as that in Example 7.8. Thus, the numbers would pass the runs-up and runs-down test. However, it can be observed that the first 20 numbers are all above the mean [(0.99 + O.OO)/2 = 0.495] and the last 20 numbers are all below the mean. Such an occurrence is highly unlikely. The previous runs analysis can be used to test for this condition, ifthe definition of a run is changed. Runs will be described as being above the mean or below the mean. A " + " sign will be used to denote an observation above the mean, and a "-" sign will denote an observation below the mean.

For example, consider the following sequence of 20 two-digit random numbers;

0.40 0.84 0.75 0.18 0.13 0.92 0.57 0.77 0.30 0.71

0.42 0.05 0.78 0.74 0.68 0.03 0.18 0.51 0.10 0.37

The pluses and minuses are as follows:

- + + - - + + + - + - - + + + - - + - -

In this case, there is a run of length one below the mean followed by a run of length two above the mean, and so on. In all. there are 11 runs, five of which are above the mean and six of which are below the mean. Let n1 and n2 be the number of individual observations above and below the mean and let b be the total number of runs. Notice that the maximum number of runs is N = n1 + n2 and the minimum number of runs is one.

Given n1 and n2. the mean — with a continuity correction suggested by Swed and Eisenhart [1943] —and the variance of b for a truly independent sequence are given by

$\mu_b = 2 n_1 n_2 / N + 1/2$ (7.6)

and

$\sigma_b^2 = 2_{n1n2} (2_{n1n2}-N) / N_2(N - 1)$ (7.7)

For either n1 or n2 greater than 20, b is approximately normally distributed. The test statistic can be formed by subtracting the mean from the number of runs and dividing by the standard deviation, or

$Z_0 = (b-(2_{n1n2}/N)-1/2) / [2_{n1n2}(2n1n2-N/N^2(N-1)]^{½}$

Failure to reject the hypothesis of independence occurs when $-z_a/2 <= Z_0 < =z_a/2-$, where a is the level of significance.

EXAMPLE 4.9

Determine whether there is an excessive number of runs above or below the mean for the sequence of numbers given in Example 7.8. The assignment of + 's and — 's results in the following:

- + + + + + + + - - - + + - + - - - - - - - + +

- - - - + + - - + - + - - + + -

The values of n1 ,n2 , and b are as follows:

$n_1 = 18$

$n_2 = 22$

$N = n_1 + n_2 = 40$

b = 17

Equations (7.6) and (7.7) are used to determine μ_b and σ_b^2 as follows:

$\mu_b = 2(18)(22)/40 + 1/2 = 20.3$

$\sigma_b^2 = 2(18)(22)[(2)(18)(22)-40] /(40)^2 (40-1)=9.54$

Since n_2 is greater than 20, the normal approximation is acceptable, resulting in a Z_0 value of

$Z_0 = 17\text{-}20.3/\sqrt{9.54} = -1.07$

Since $Z_{0.025} = 1.96$ the hypothesis of independence cannot be rejected on the basis of this test

3. Runs test: length of runs. Yet another concern is the length of runs. As an example of what might occur, consider the following sequence of numbers

0.16, 0.27, 0.58, 0.63, 0.45, 0.21, 0.72, 0.87, 0.27, 0.15, 0.92, 0.85,...

Assume that this sequence continues in a like fashion: two numbers below the mean followed by two numbers above the mean. A test of runs above and below the mean would detect no departure from independence. However, it is to be expected that runs other than of length two should occur.

Let Y, be the number of runs of length i in a sequence of N numbers. For an independent sequence, the expected value of Yj for runs up and down is given by

$E(Yi) = 2/(i+3)![\ N(i^2+3i+1)-(i^3+3i^2-i-4)\ ]\ ,i <= N - 2$ (7.8)

$E(Yi) =2/N!\ i = N - 1$ (7.9)

For runs above and below the mean, the expected value of y, is approximately given by

$E(Yi) = N\ w_i / E(I\ ),\ N>20$ (7.10)

where w_i the approximate probability that a run has length i, is given by

$w_i=(\ n_1\ /N)^I\ (n_2/N) + (n_1/N)(n_2/N)^I\ N>20$ (7.11)

and where E(I) , the approximate expected length of a run, is given by

$E(I\ )= n_1\ / n_2 + n_2/\ n_1\ N>20$ (7.12)

The approximate expected total number of runs (of all lengths) in a sequence of length N, E(A), is given by

$E(A)=N/\ E(I)\ N>20$ (7.13)

The appropriate test is the chi-square test with Oi, being the observed number of runs of length i. Then the test statistic is

$X_0^2=\Sigma_{\ i\ =\ 1}^{\ L}\ [\ Oi - E(Yi)^2\ ]\ /\ E(Yi)$

where L = N - 1 for runs up and down and L = N for runs above and below the mean. If the null hypothesis of independence is true, then X_0^2 is approximately chi-square distributed with L — 1 degrees of freedom.

EXAMPLE 4.10

Given the following sequence of numbers, can the hypothesis that the numbers are independent be rejected on the basis of the length of runs up and down at

a =0.05?

0.30 0.48 0.36 0.01 0.54 0.34 0.96 0.06 0.61 0.85

0.48 0.86 0.14 0.86 0.89 0.37 0.49 0.60 0.04 0.83

0.42 0.83 0.37 0.21 0.90 0.89 0.91 0.79 0.57 0.99

0.95 0.27 0.41 0.81 0.96 0.31 0.09 0.06 0.23 0.77

0.73 0.47 0.13 0.55 0.11 0.75 0.36 0.25 0.23 0.72

0.60 0.84 0.70 0.30 0.26 0.38 0.05 0.19 0.73 0.44

For this sequence the +'s and —'s are as follows

+ - - + - + - + + - + - + + - + + - + - + - - + - + - - + - - + + + - - - + + - - - + - + - - - + - + - - - + - + + -

The length of runs in the sequence is as follows:

1,2,1,1,1,1,2,1,1,1,2,1,2,1,1,1,1,2,1,1, 1,2,1,2,3,3,2,3,1,1,1,3,1,1,1,

3,1,1,2,1

The number of observed runs of each length is as follows:

Run length ,i	1	2	3
Observe Run **0i**	26	9	5

The expected numbers of runs of lengths one, two, and three are computed from Equation (7.8) as

$$E(Yi) = 2/4![60(1 + 3 + 1) - (1 + 3 - 1 - 4)]= 25.08$$

$$E(Y2) = 2/5![60(4 + 6 + 1) - (8 + 12 - 2- 4)] = 10.77$$

$$E(Y3) =2/6![60(9 + 9 + 1) – (27 + 27 - 3 – 4)] =3.04$$

The mean total number of runs (up and down) is given by Equation (7.4) as

$\mu a = 2(60)-1/3=39.67$

Thus far, the E(Yi) for i = 1, 2, and 3 total 38.89. The expected number of runs of length 4 or more is the difference $\mu a - \Sigma^{3}_{i=1} E(Yi)$ or 0.78

Table 7.4. Length of Runs Up and Down: x^2 Test

Rim Length, i	*Observed Number of Runs, Oj*	*Expected Number of Runs, E(Y,)*	*[0, - E(Y,)]² / E(Yi)*
1	26	25.08	0.03
2	9 }	10.77 }	
>=3	5 } **14**	3.82 } **14.59**	0.02
	40	39.67	0.05

Tests for Autocorrelation

The tests for autocorrelation are concerned with the dependence between numbers in a sequence. As an example, consider the following sequence of numbers:

0.12 0.01 0.23 0.28 0.89 0.31 0.64 0.28 0.83 0.93

0.99 0.15 0.33 0.35 0.91 0.41 0.60 0.27 0.75 0.88

0.68 0.49 0.05 0.43 0.95 0.58 0.19 0.36 0.69 0.87

From a visual inspection, these numbers appear random, and they would probably pass all the tests presented to this point. However, an examination of the 5th, 10th, 15th (every five numbers beginning with the fifth), and so on. indicates a very large number in that position. Now, 30 numbers is a rather small sample size to reject a random-number generator, but the notion is that numbers in the sequence might be related. In this particular section, a method for determining whether such a relationship exists is described. The relationship would not have to

be all high numbers. It is possible to have all low numbers in the locations being examined, or the numbers may alternately shift from very high to very low.

The test to be described below requires the computation of the autocorrelation between every m numbers (m is also known as the lag) starting with the ith number. Thus, the autocorrelation p,m between the following numbers would be of interest: /?,-, Rj+m, Ri+2m, • • •, Ri+(M+\)m- The value M is the largest integer such that / + (M + l)m < N, where N is the total number of values in the sequence. (Thus, a subsequence of length M + 2 is being tested.)

Since a nonzero autocorrelation implies a lack of independence, the following two tailed test is appropriate:

H0: ρim= 0

Hi: ρim ≠ 0

For large values of M, the distribution of the estimator of ρim denoted ρ^im is approximately normal if the values Ri, Ri+m, Ri+2m,…….Ri+(M+1)m are un-correlated. Then the test statistic can be formed as follows:

Z0 = ρ^im / σ ρ^im

which is distributed normally with a mean of zero and a variance of 1, under the assumption of independence, for large M.

The formula for ρ^im in a slightly different form, and the standard deviation of the estimator, σ ρ^im are given by Schmidt and Taylor [1970] as follows:

ρ^im = 1/ M +1**[** ΣM k=0 Ri +km Ri+(k+1)m **]** - 0.25

σ ρ^im = √ (13M + 7) / 12(M +1)

After computing Z0, do not reject the null hypothesis of independence if - **za/2** <= **Z0** <= **za/2**, where *a* is the level of significance.

If ρim > 0, the subsequence is said to exhibit positive autocorrelation. In this case, successive values at lag *m* have a higher probability than expected of being close in value (i.e., high random numbers in the subsequence followed by high, and low followed by low). On the other hand, if ρim < 0, the subsequence is exhibiting negative autocorrelation, which means that low random numbers tend to be followed by high ones, and vice versa. The desired property of independence, which implies zero autocorrelation, means that there is no discernible relationship of the nature discussed here between successive, random numbers at lag *m*.

EXAMPLE 4.12

Test whether the 3rd, 8th, 13th, and so on, numbers in the sequence at the beginning of this section are autocorrelated. (Use $a = 0.05$.) Here, $i = 3$ (beginning with the third number), $m = 5$ (every five numbers), $N = 30$ (30 numbers in the sequence), and $M = 4$ (largest integer such that $3 + (M+1)5 < 30$). Then,

ρ^35 = 1/ 4 + 1**[** (0.23)(0.28) + (0.28)(0.33) + (0.33)(0.27) + (0.27)(0.05) + (0.05)(0.36) **]** = - 0.1945

and σ ρ^35= √ (13(4) + 7) / 12(4 + 1) = 0.1280

Then, the test statistic assumes the value

Z0 = -0.1945/0.1280 = -1.516

Now, the critical value is

Z0.025 = 1.96

Therefore, the hypothesis of independence cannot be rejected on the basis of this test.

It can be observed that this test is not very sensitive for small values of M, particularly when the numbers being tested are on the low side. Imagine what would happen if each of the entries in the foregoing computation of ρ^im were equal to zero. Then, ρ^im would be equal to —0.25 and the calculate would have the value of —1.95, not quite enough to reject the hypothesis of independence.

Many sequences can be formed in a set of data, given a large value of *N*. For example, beginning with the first number in the sequence, possibilities include

1) the sequence of all numbers,

(2) the sequence formed from the first. third, fifth,..., numbers,

(3) the sequence formed from the first, fourth, numbers, and so on. If *a* — 0.05, there is a

probability of 0.05 of rejecting a true hypothesis. If 10 independent sequences are examined, the probability of finding no significant autocorrelation, by chance alone, is (0.95)10 or 0.60. Thus, 40% of the time significant autocorrelation would be detected when it does not exist. If *a* is 0.10 and 10 tests are conducted, there is a 65% chance of finding autocorrelation by chance alone. In conclusion, when "fishing" for autocorrelation, upon performing numerous tests, autocorrelation may eventually be detected, perhaps by chance alone, even when no autocorrelation is present.

Gap Test

The gap test is used to deter.nine the significance of the interval between the recurrences of the same digit. A gap of length *x* occurs between the recurrences of some specified digit.

The following example illustrates the length of gaps associated with the digit 3:

4, 1, 3, 5, 1, 7. 2. 8. 2. 0, 7. 9. 1. 3. 5, 2, 7, 9. 4. 1. 6. 3

3, 9. 6, 3, 4. 8, 2. 3, 1, 9, 4. 4, 6. 8. 4, 1, 3. 8. 9. 5. 5. 7

3, 9, 5, 9. 8, 5. 3. 2, 2, 3, 7. 4, 7. 0. 3. 6. 3, 5, 9. 9. 5. 5

5, 0, 4. 6. 8. 0, 4. 7. 0, 3. 3, 0, 9, 5. 7, 9. 5. 1. 6. 6. 3. 8

8, 8, 9, 2, 9. 1. 8. 5, 4. 4. 5, 0, 2. 3, 9, 7. 1. 2. 0. 3, 6. 3

To facilitate the analysis, the digit 3 has been underlined. There are eighteen 3's in the list. Thus, only 17 gaps can occur. The first gap is of length 10. the second gap is of length 7, and so on. The frequency of the gaps is of interest. The probability of the first gap is determined as follows:

10 of these terms

$$P(\text{gapof } l0) = P(\text{no3})\text{---}P(\text{no3})P(3) = (0.9)10\ (0.1) \qquad (7.12)$$

since the probability that any digit is not a 3 is 0.9, and the probability that any digit is a 3 is 0.1. In general,

P(t followed by exactly *x* non-r digits) = $(0.9)^x$ (0.1), X = 0.1.2..

The theoretical frequency distribution for randomly ordered digits is given by

$P(\text{gap} <= x) = F(x) = 0.1 \Sigma\ ^{x}_{n=0} (0.9)^n = 1 - 0.9^{x+1}$

The procedure for the test follows the steps below. When applying the test to random numbers, class intervals such as [0, 0.1), [0.1,0.2),. . . play the role of random digits.

Step 1. Specify the cdf for the theoretical frequency distribution given by Equation (7.14) based on the selected class interval width.

Step2.Arrange the observed sample of gaps in a cumulative distribution with these same classes.

Step 3. Find D, the maximum deviation between *F(x)* and *SN(X)* as in Equation (7.3).

Step 4. Determine the critical value, Da, from Table A.8 for the specified value of *a* and the sample size *N*.

Step 5. If the calculated value of *D* is greater than the tabulated value of Da, the null hypothesis of independence is rejected.

EXAMPLE 4.13

Based on the frequency with which gaps occur, analyze the 110 digits above to test whether they are independent. Use *a* = 0.05. The number of gaps is given by the number of data values minus the number of distinct digits, or 110 —10 = 100 in the example. The number of gaps associated with the various digits are as follows:

Digit	0	1	2	3	4	5	6	7	8	9
Number of gaps	7	8	8	17	10	13	7	8	9	13

The gap test is presented in Table 7.6. The critical value of D is given by

D0.05 = 1.36 / √ 100 = 0.136

Since D = max |F(x) - SN(x)| = 0.0224 is less than D0.05 do not reject the hypothesis of independence on the basis of this test.

Table 7.6. Gap-Test Example

Gap Length	Frequency	Relative Frequency	Cumulative Relative Frequency	F(x)	$\lvert F(x)-s_N(x)\rvert$
0-3	35	0.35	0.35	0.3439	0.0061
4-1	22	0.22	0.57	0.5695	0.0005
8-11	17	0.17	0.74	0.7176	0.0224
12-15	9	0.09	0.83	0.8147	0.0153
16-19	5	0.05	0.88	0.8784	0.0016
20-23	6	0.06	0.94	0.9202	0.0198
24-27	3	0.03	0.97	0.9497	0.0223
28-31	0	0.0	0.97	0.9657	0.0043
32-35	0	0.0	0.97	0.9775	0.0075
36-39	2	0.02	0.99	0.9852	0.0043
40-43	0	0.0	0.99	0.9903	0.0003
44-47	1	0.01	1.00	0.9936	0.0064

Poker Test

The poker test for independence is based on the frequency with which certain digits are repeated in a series of numbers. The following example shows an unusual amount of repetition:

0.255, 0.577, 0.331, 0.414, 0.828, 0.909, 0.303,0.001, ...

In each case, a pair of like digits appears in the number that was generated. In three-digit numbers there are only three possibilities, as follows:

1. The individual numbers can all be different.

2. The individual numbers can all be the same.

3. There can be one pair of like digits.

The probability associated with each of these possibilities is given by the following

P(three different digits) = *P*(second different from the first) x P(third different from the first and second) = (0.9)(0.8) = 0.72

P(three like digits) = P(second digit same as the first) x *P* (third digit same as the first) = (0.1)(0.1) = 0.01

P(exactly one pair) = 1 - 0.72 - 0.01 = 0.27

Alternatively, the last result can be obtained as follows:

P(exactly one pair) = $\binom{3}{2}$ (0.1)(0.9) = 0.27

The following example shows how the poker test (in conjunction with the chi-square test) is

used to ascertain independence.

EXAMPLE 4.14

A sequence of 1000 three-digit numbers has been generated and an analysis indicates that 680 have three different digits, 289 contain exactly one pair of like digits, and 31 contain three like digits. Based on the poker test, are these numbers independent? Let a = 0.05. The test is summarized in Table 7.7.

The appropriate degrees of freedom are one less than the number of class intervals. Since $47.65 > X^2_{0.05.2} = 5.99$. the independence of the numbers is rejected on the basis of this test.

Table 7.7. Poker-Test Results

Combination, i	*Observed Frequency,* O,	*Expected Frequency,* £,	$\frac{(0, - E i)^2}{E,}$
Three different digits	680	720	2.22
Three like digits	31	10	44.10
Exactly one pair	289	270	1.33
	1000	1000	47.65

RANDOM - VARIATE GENERATION

INTRODUCTION :

This chapter deals with procedures for sampling from a variety of widely used continuous and discrete distributions. Here it is assumed that a distribution has been completely specified, and ways are sought to generate samples from this distribution to be used as input to a simulation model. The purpose of the chapter is to explain and illustrate some widely used techniques for generating random variates, not to give a stateof-the-art survey of the most efficient techniques.

TECHNIQUES:

- **INVERSE TRANSFORMATION TECHNIQUE**
- **ACCEPTANCE-REJECTION TECHNIQUE**

All these techniques assume that a source of uniform (0,1) random numbers is available R1,R2….. where each R1 has probability density function

pdf fR(X)= 1 , 0<= X <=1

0 , otherwise

cumulative distribution function

cdf fR(X)= 0 , X<0

X , 0<= X <= 1

1 , X>1

The random variables may be either discrete or continuous.

5.6**Inverse Transform Technique :**

The inverse transform technique can be used to sample from exponential, the uniform, the Weibull, and the triangular distributions and empirical distributions. Additionally, it is the underlying principle for sampling from a wide variety of discrete distributions. The technique will be explained in detail for the exponential distribution and then applied to other distributions. It is the most straightforward, but always the most efficient., technique computationally.

EXPONENTIAL DISTRIBUTION :

The exponential distribution, has probability density function (pdf) given by

$f(X)= \lambda e^{-\lambda x}$, $x \geq 0$

0, $x < 0$

and cumulative distribution function (cdf) given by

$f(X) = \int_{-\infty}^{x} f(t)\, dt =$ $1 - e^{-\lambda}x$, $x \geq 0$

0, $x < 0$

The parameter can be interpreted as the mean number of occurrences per time unit.

For example, if interarrival times Xi , X2, X3, . . . had an exponential distribution with rate , then could be interpreted as the mean number of arrivals per time unit, or the arrival rate| Notice that for any j

$E(Xi)= 1/\lambda$

so that is the mean interarrival time. The goal here is to develop a procedure for generating values X1,X2,X3, ….. which have an exponential distribution.

The inverse transform technique can be utilized, at least in principle, for any distribution. But it is most useful when the cdf. F(x), is of such simple form that its inverse, F , can be easily

computed. A step-by-step procedure for the inverse transform technique illustrated by me exponential distribution, is as follows:

Step 1. Compute the cdf of the desired random variable X. For the exponential distribution, the cdf is F(x) = 1 — e , x > 0.

Step 2. Set F(X) = R on the range of X. For the exponential distribution, it becomes $1 - e^{-\lambda X} = R$ on the range x >=0. Since X is a random variable (with the exponential distribution in this case), it follows that 1 - is also a random variable, here called R. As will be shown later, R has a uniform distribution over the interval (0,1).,

Step 3. Solve the equation F(X) = R for X in terms of R. For the exponential distribution, the solution proceeds as follows:

$$1 - \quad e^{-\lambda}x = R$$

$$e^{-\lambda}x = 1 - R$$

$$-\lambda X = \ln(1 - R)$$

$$x = -1/\lambda \ln(1 - R) \qquad (5.1)$$

Equation (5.1) is called a random-variate generator for the exponential distribution. In general, Equation (5.1) is written as X=F-1(R). Generating a sequence of values is accomplished through steps 4.

Step 4. Generate (as needed) uniform random numbers R1, R2, R3,... and compute the desired random variates by

$$Xi = F\text{-}1 (Ri)$$

For the exponential case, F (R) = (-1/λ)ln(1- R) by Equation (5.1), so that

$$Xi = -1/\lambda \ln (1 - Ri) \quad (5.2)$$

for i = 1,2,3,.... One simplification that is usually employed in Equation (5.2) is to replace 1 – Ri by Ri to yield

$$Xi = -1/\lambda \ln Ri \quad (5.3)$$

which is justified since both Ri and 1- Ri are uniformly distributed on (0,1).

Table 5.1 Generation of Exponential Variates X, with Mean 1, Given Random Numbers Ri,

I	1	2	3	4	5
Ri	0.1306	0.0422	0.6597	0.7965	0.7696
Xi	0.1400	0.0431	1.078	1.592	1.468

Table 5.1 gives a sequence of random numbers from Table A.1 and the computed exponential variates, Xi, given by Equation (5.2) with a value of = 1. Figure 5.1(a) is a histogram of 200 values, R1,R2,…R200 from the uniform distrubition and figure 5.1(b)

Figure 5.1. (a) Empirical histogram of 200 uniform random numbers; (b) empirical histogram of 200 exponential variates; (c) theoretical uniform density on (0,1); (d) theoretical exponential density with mean 1.

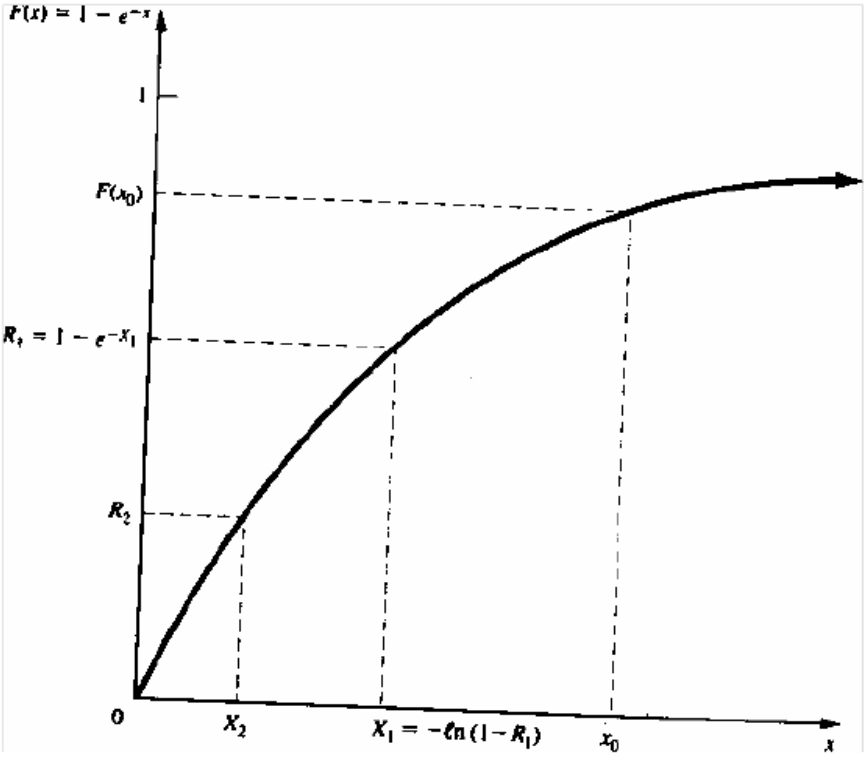


Figure 5.2. Graphical view of the inverse transform technique.

and Figure 5.1(b) is a histogram of the 200 values, X1,X2,…,X200, computed by Equation (5.2). Compare these empirical histograms with the theoretical density functions in Figure 5.1(c) and (d). As illustrated here, a histogram is an estimate of the underlying density function. (This fact is used in Chapter 9 as a way to identify distributions.)

Figure 5.2 gives a graphical interpretation of the inverse transform technique. The cdf shown is F(x) = 1- e an exponential distribution with rate . To generate a valueX1 with cdf F(X), first a random number Ri between 0 and 1 is generated, a horizontal line is drawn from R1 to the graph of the cdf, then a vertical line is dropped to the x -axis to obtain X1, the desired result. Notice the inverse relation between R1 and X1, namely

$$Ri = 1 - e^{-X1}$$

And

$X_1 = -\ln (1 - R1)$

In general, the relation is written as

$$R_1 = F (X_1)$$

and

$$X_1 = F^{-1} (R1)$$

Why does the random variable X1 generated by this procedure have the desired distribution? Pick a value JCQ and compute the cumulative probability

$$P(X1 < x0) = P(R1 < F(x0)) = F(X0)$$

To see the first equality in Equation (8.4), refer to Figure 5.2, where the fixed numbers x0 and F(X0) are drawn on their respective axes. It can be seen that X1 < xo when and only when R1 < F(X0). Since 0 < F(xo) < 1, the second equality in Equation (8.4) follows immediately from the fact that R\ is uniformly distributed on (0,1). Equation (8.4) shows that the cdf of X1 is F; hence, X1 has the desired distribution.

Uniform Distribution :

Consider a random variable X that is uniformly distributed on the interval [a, b]. A reasonable guess for generating X is given by

$$X = a + (b - a)R \qquad (5.5)$$

[Recall that R is always a random number on (0,1). The pdf of X is given by

$$f(x) = \begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

The derivation of Equation (5..5) follows steps 1 through 3 of Section 5.1.1:

Step 1. The cdf is given by

$$F(x) = \begin{cases} 0, & x < a \\ (x - a)/(b - a), & a \le x \le b \\ 1, & x > b \end{cases}$$

Step 2. Set F(X) = (X - a)/(b -a) = R

Step 3. Solving for X in terms of R yields X = a + (b — a)R, which agrees with Equation (5.5).

Discrete Distribution

All discrete distributions can be generated using the inverse transform technique, either numerically through a table-lookup procedure, or in some cases algebraically with the final generation scheme in terms of a formula. Other techniques are sometimes used for certain distributions, such as the convolution technique for the binomial distribution. Some of these methods are discussed in later sections. This subsection gives examples covering both empirical distributions and two of the standard discrete distributions, the (discrete) uniform and the geometric. Highly efficient table-lookup procedures for these and other distributions are found in Bratley, Fox, and Schrage [1987] and Ripley [1987].

Table 5.5. Distribution of Number of Shipments, X

X	PM	F(x)
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00

Example 1 (An Empirical Discrete Distribution) :

At the end of the day, the number of shipments on the loading dock of the IHW Company (whose main product is the famous, incredibly huge widget) is either 0, 1, or 2, with observed relative frequency of occurrence of 0.50, 0.30, and 0.20, respectively. Internal consultants have been asked to develop a model to improve the efficiency of the loading and hauling operations, and as part of this model they will need to be able to generate values, X, to represent the number of shipments on the loading dock at the end of each day. The consultants decide to model X as a discrete random variable with distribution as given in Table 5.5 and shown in Figure 5.6.
The probability mass function (pmf), P(x) is given by

$$p(0) = P(X = 0) = 0.50$$

$$p(1) = P(X = 1) = 0.30$$

$$p(2) = P(X = 2) = 0.20$$

and the cdf, $F(x) = P(X < x)$, is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.5, & 0 \le x < 1 \\ 0.8, & 1 \le x < 2 \\ 1.0 & 2 \le x \end{cases}$$

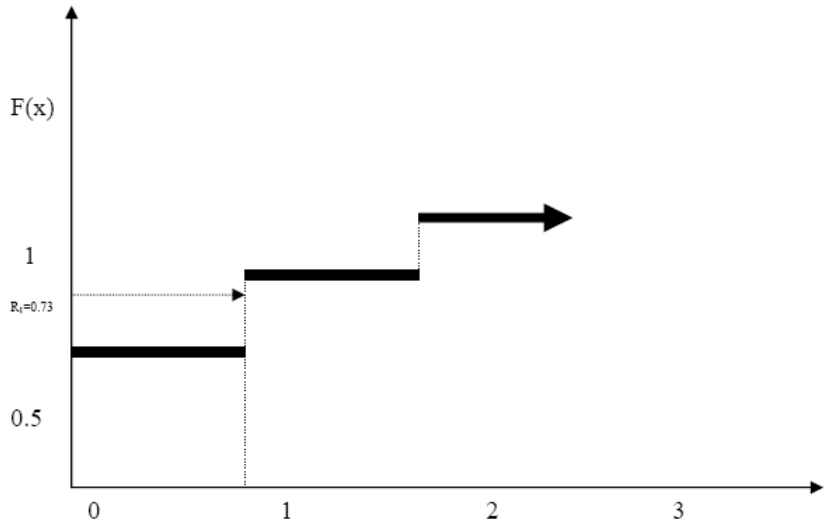


Figure 5.6. The cdf of number of shipments, X.

Table 5.6. Table for Generating the Discrete Variate

X

I	*Input, n*	*Output, Xi*
1	0.50	0
2	0.80	1
3	1.00	2

Recall that the cdf of a discrete random variable always consists of horizontal line segments with jumps of size p(x) at those -points, x, which the random variable can assume. For example, in Figure 8.6 there is a jump of size = 0.5 at x = 0, of size p(l)=0.3 at x=1, and of size p(2) = 0.2 at x=2.

For generating discrete random variables, the inverse transform technique becomes a table-lookup procedure, but unlike the case of continuous variables, interpolation is not required. To illustrate the procedure, suppose that R1 = 0.73 is generated. Graphically, as illustrated in Figure 8.6, first locate R1 = 0.73 on the vertical axis, next draw a horizontal line segment until it hits a "jump" in cdf, and then drop a perpendicular to the horizontal axis to get the generated variate. Here R1 = 0.73 is transformed to X1 = 1. This procedure is analogous to the procedure used for empirical continuous distributions, except that the final step of linear interpolation is eliminated.

The table-lookup procedure is facilitated by construction of a table such as table 5.6. When R1 == 0.73 is generated, first find the interval in which R1 lies. In general, for R = R1, if

$$F(x_{i-1}) = r_{i-1} < R \leq r_i = F(x_i) \tag{5.13}$$

then set X1 = xi. Here ro = 0, x0 = , while x1,x2,….,xn are the possible values of the random variable, and rk = p(x1) +….. + p(xk), k = 1,2,..., n. For this example, n = 3, x1 = 0, x2 = 1, x3 = 2, and hence r1 = 0.5, r2 = 0.8, and r3 = 1.0. (Notice that rn = 1.0 in all cases.)

Since n = 0.5 < R1 = 0.73 < r2=0.8, set X1 = x2 = 1. The generation scheme is summarized as follows:

$$X = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 < R \le 0.8 \\ 2, & 0.8 < R \le 1.0 \end{cases}$$

Example 8.4 illustrates the table-lookup procedure, while the next example illustrates an algebraic approach that can be used for certain distributions.

Example 2 (A Discrete Uniform Distribution) :

Consider the discrete uniform distribution on {1,2,..., k} with pmf and cdf given by

p(x) = 1/k, x = 1,2, . ..k,

and

$$F(x) = \begin{cases} 0, & x < 1 \\ 1/k, & 1 \le x < 2 \\ 2/k, & 2 \le x < 3 \\ \vdots & \\ k-1/k, & k-1 \le x < k \\ 1, & k \le x \end{cases}$$

Let xi=i and ri = p(l) + ……. + p(xi) —F(xi) = i/k for i=1,2,..., k. Then by using Inequality (5.13) it can be seen that if the generated random number R satisfies

$$R\ i\text{-}1 = i\text{-}1 / k < R \leq ri = i/k \quad (5.14)$$

then X is generated by setting X = i . Now, Inequality (5.14) can be solved for j:

$$i - 1 < Rk \leq i$$

$$Rk \leq i < Rk + 1 \quad (5.15)$$

Let [y] denote the smallest integer > y. For example, [7.82] = 8, [5.13] = 6 and [-1,32] = -1. For y > 0, [y] is a function that rounds up. This notation and Inequality (5.15) yield a formula for generating X, namely

$$X = \lceil R\ k \rceil$$

For example, consider generating a random variate X, uniformly distributed on {1,2,..., 10}. The variate, X, might represent the number of pallets to be loaded onto a truck. Using Table A.I as a source of random numbers, R, and Equation (5.16) with k = 10 yields

$R_1 = 0.78, \quad X_i = [7.8] = 8$

$R_2 = 0.03, \quad X_2 = [0.3] = 1$

$R_3 = 0.23, \quad X_3 = [2.3] = 3$

$R_4 = 0.97, \quad X_4 = [9.7] = 10$

The procedure discussed here can be modified to generate a discrete uniform random variate with any range consisting of consecutive integers. Exercise 13 asks the student to devise a procedure for one such case.

Example 3 (The Geometric Distribution)

Consider the geometric distribution with pmf

$$p(x) = p\ (\ l - p\ )^{x}\ ,\ x = 0,1,2,...$$

where $0 < p < 1$. Its cdf is given by

$$F(x) = \Sigma^{x}_{j=0}\, p(1-p)^{j}$$

$= p\{1- (1- p)^{x+1} \} / 1 – (1 – p)$

$= 1 – (1 – p)^{x+1}$

for x = 0, 1, 2, ... Using the inverse transform technique [i.e., Inequality (5.13)], recall that a geometric random variable X will assume the value x whenever

$$F(x - 1) = 1 - (1 –p)x < R < 1 - (1 – p)\ x+1 = F(x) \quad (5.19)$$

where R is a generated random number assumed 0 < R <1. Solving Inequality (5.19) for x proceeds as follows:

$(1-p)^{x+1} \leq 1 – R < (1 – p)^{x}$

$(x + 1)\ln (1 – p) \leq \ln (1 – R) < x \ln(1 – p)$

But 1 – p < 1 implies that ln(1-p) < 0. so that

$$\text{Ln} (1 –R) / \ln (1 –p) -1 \leq x < \ln(1 – R) / \ln (1 – p) \qquad (5.20)$$

Thus, X=x for that integer value of x satisfying Inequality (5.20) or in brief using the round-up

function ⌜ ·⌝

$$X = \ln (1 – R) / \ln (1 – p) – 1 \quad (5.21)$$

Since p is a fixed parameter, let = —l/ £ n(l — p). Then > 0 and, by Equation (5.21), X = [-]. By Equation (5.1), is an exponentially distributed random variable with mean , so that one way of generating a geometric variate with parameter p is to generate (by any method) an exponential variate with parameter, subtract one, and round up.Occasionally, a geometric variate X is needed which can assume values {q, q + 1, q + 2,...} with pmf p(x) = p(1 - p) (x = q, q +1,...). Such a variate, X can be generated,

using Equation (5.21), by

$$X = q + \ln (1-R)/ \ln(1 – p) -1$$

One of the most common cases is q = 1.

5.7 Acceptance-Rejection Technique :

Suppose that an analyst needed to devise a method for generating random variates, X, uniformly distributed between . and 1. One way to proceed would be to follow these steps:

Step1 : Generate a random number R.

Step 2a.: If R > 1/4, accept X = /?, then go to step 3.

Step 2b.: If R < 1/4, reject /?, and return to step 1.

Step 3.: If another uniform random variate on [1/4,1] is needed, repeat the procedure beginning at step 1. If not, stop.

Each time step 1 is executed, a new random number R must be generated. Step 2a is an ―acceptance‖ and step 2b is a "rejection" in this acceptancerejection technique. To summarize the technique, random variates (R) with some distribution (here uniform on [0,1]) are generated until some condition (R > 1/4) is satisfied. When the condition is finally satisfied, the desired random variate, X (here uniform on [1/4,1]), can be computed (X = R). This procedure can be shown to be correct by recognizing that the accepted values of R are conditioned values; that is, R itself does not have the desired distribution, but R conditioned on the event {R > 1/4} does have the desired distribution.

To show this, take 1/4 < a < b < 1; then

$$P (a < R \leq b \mid . \leq R \leq 1) = P (a < R \leq b) / P (. \leq R \leq 1) = b – a / 3/4 \qquad (5.28)$$

which is the correct probability for a uniform distribution on [1/4, 1]. Equation (5.28) says that the probability distribution of /?, given that R is between 1/4 and 1 (all other values of R are thrown out), is the desired distribution. Therefore, if 1/4 < R < 1, set X = R.

Poisson Distribution :

A Poisson random variable, N, with mean a > 0 has pmf

$$p(n) = P(N = n) = e^{-\alpha} \alpha^n / n! , \qquad n = 0,1,2,…….$$

but more important, N can be interpreted as the number of arrivals from a Poisson arrival process in one unit of time. Recall that the inter- arrival times, A1, A2,... of successive customers are exponentially distributed with rate a (i.e., a is the mean number of arrivals per unit time); in addition, an exponential variate can be generated by Equation (5.3). Thu/there is a relationship between the (discrete) Poisson distribution and the (continuous) exponential distribution, namely

$$N = n \qquad (5.29)$$

if and only if $A1 + A2 + ……+ An \leq 1 < A1 + ….. + An + An+1$ (5.30)

Equation (5.29)/N = n, says there were exactly n arrivals during one unit of time; but relation (8.30) says that the nth arrival occurred before time 1 while the (n + l)st arrival occurred after

time l) Clearly, these two statements are equivalent. Proceed now by generating exponential interarrival times until some arrival, say n + 1, occurs after time 1; then set N = n.

For efficient generation purposes, relation (5.30) is usually simplified by first using Equation (5.3), Ai= (—1/ α)ln Ri, to obtain

$$\Sigma_{i=1}^{n} - 1/\alpha \ln Ri \leq 1 < \Sigma^{n+1}_{i=1} -1/\alpha \ln Ri$$

Next multiply through by reverses the sign of the inequality, and use the fact that a sum of logarithms is the logarithm of a product, to get

$$\ln \Pi^{n}_{i=1} Ri = \Sigma^{n}_{i=1} \ln Ri \geq - \alpha > \Sigma^{n+1}_{i=1} \ln Ri = \ln \Pi^{n+1}_{i=1} Ri$$

Finally, use the relation elnx=x for any number x to obtain

$$\Pi^{n}_{i=1} Ri \geq e^{-\alpha} > \Pi^{n+1}_{i=1} Ri \qquad (8.31)$$

which is equivalent to relation (8.30).The procedure for generating a Poisson random variate, N, is given by the following steps:

Step 1. Set n = 0, P = 1.

Step 2. Generate a random number R_n+i and replace P by P •R_n+i.

Step 3. If P < e^{-0}, then accept N = n. Otherwise, reject the current n, \ increase n

by one, and return to step 2.

Notice that upon completion of step 2, P is equal to the rightmost expression in relation (5.31). The basic idea of a rejection technique is again exhibited; if P > $e^{\sim a}$ in step 3, then n is rejected and the generation process must proceed through at least one more trial.

How many random numbers will be required, on the average to generate one poisson variate, N? if N=n, then n+1 random numbers are required so the average number is given by

$$E(N+1) = \alpha + 1$$

Which is quite large if the mean , alpha, of the poisson distribution is large.

Example 4 :

Generate three Poisson variates with mean a =-02. First compute £"" = e^°'2 = 0.8187. Next get a sequence of random numbers R from Table A.I and follow steps 1 to 3 above:

Step l. Set n = 0, P = 1.

Step 2. R1 = 0.4357, P = 1 • R1 = 0.4357.

Step 3. Since P = 0.4357 < e^{-b} = 0.8187, accept N = 0.

Step 1-3. (Ri =0.4146 leads to N = 0.)

n	R_{n+1}	p	accept/reject	Result
0	0.4357	0.4357	P<$e^{-\alpha}$ (accept)	N=0
0	0.4146	0.4146	P<$e^{-\alpha}$ (accept)	N=0
0	0.8353	0.8353	P≥$e^{-\alpha}$ (reject)	
1	0.9952	0.8313	P≥ $e^{-\alpha}$ (reject)	
2	0.8004	0.6654	p < $e^{-\alpha}$ (accept)	N=2

Step l. Set n = 0, P = 1.

Step 2. R1=0.8353, P = 1 - R1 = 0.8353.

Step 3. Since P > e^{-b}, reject n = 0 and

return to step 2 with n = 1.

Step 2. R2 = 0.9952, P = R_1R_2 = 0.8313.)

Step 3. Since P > e^{-b}, reject n = 1 and return to step 2 with n = 2.

Step 2. R3 = 0.8004, P = $R_1R_2R_3$ = 0.6654.

Step 3. Since P < e^{-b}, accept N = 2.

five random numbers, to generate three Poisson variates here (N = 0, and N = 2), but in the long

run to generate, say, 1000 Poisson variates = 0.2 it would require approximately 1000 (a +1) or 1200 random numbers.

UNIT 6

INPUT MODELING

- Input data provide the driving force for a simulation model. In the simulation of a queuing system, typical input data are the distributions of time between arrivals and service times.
- For the simulation of a reliability system, the distribution of time-to=failure of a component is an example of input data.

There are **four steps** in the development of a useful model of input data:

• *Collect data from the real system of interest*. This often requires a substantial time and resource commitment. Unfortunately, in some situations it is not possible to collect data.

• *Identify a probability distribution to represent the input process*. When data are available, this step typically begins by developing a frequency distribution, or histogram, of the data.

• *Choose parameters that determine a specific instance of the distribution family*. When data are available, these parameters may be estimated from the data.

• *Evaluate the chosen distribution and the associated parameters for good-of-fit.* Goodness-of-fit may be evaluated informally via graphical methods, or formally via statistical tests. The chisquare and the Kolmo-gorov-Smirnov tests are standard goodness-of-fit tests. If not satisfied that the chosen distribution is a good approximation of the data, then the analyst returns to the second step, chooses a different family of distributions, and repeats the procedure. If several iterations of this procedure fail to yield a fit between an assumed distributional form and the collected data.

6.1 Data Collection

The following suggestions may enhance and facilitate data collection, although they are not all – inclusive.

1. *A useful expenditure of time is in planning.* This could begin by a practice or pre observing session. Try to collect data while pre observing.

2. *Try to analyze the data as they are being collected.* Determine if any data being collected are useless to the simulation. There is no need to collect superfluous data.

3. *Try to combine homogeneous data sets.* Check data for homogeneity in successive time periods and during the same time period on successive days.

4. *Be aware of the possibility of data censoring, in which a quantity of interest is not observed in its entirety.* This problem most often occurs when the analyst is interested in the time required to complete some process (for example, produce a part, treat a patient, or have a component fail), but the process begins prior to, or finishes after the completion of, the observation period.

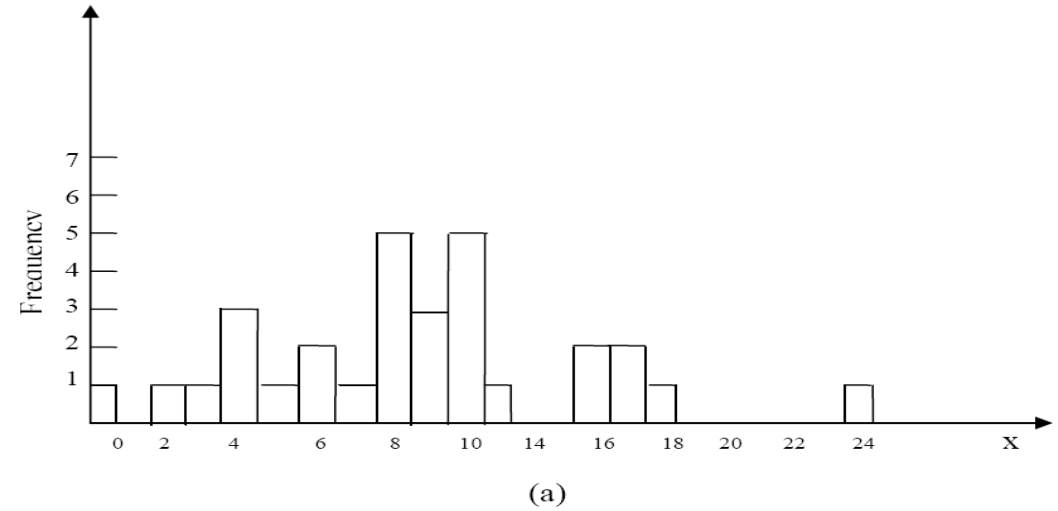
5. *To determine whether there is a relationship between two variables, build a scatter diagram.*

6. *Consider the possibility that a sequence of observations which appear to be independent may possess autocorrelation*. Autocorrelation may exist in successive time periods or for successive customers.

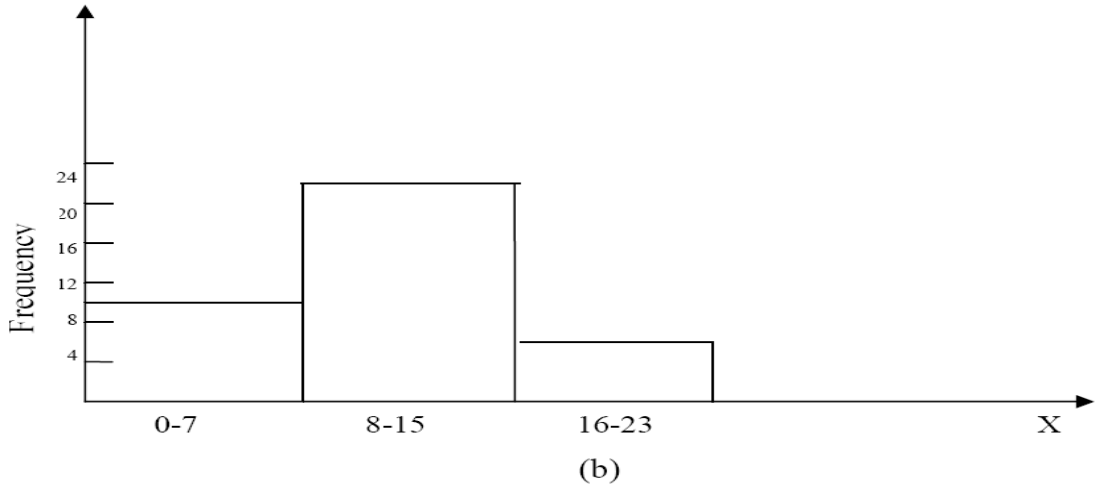
7. *Keep in mind the difference between input data and output or performance data, and be sure to collect input data*. Input data typically represent the uncertain quantities that are largely beyond the control of the system and will not be altered by changes made to improve the system.

Example 6.1 (The Laundromat)

• As budding simulation students, the first two authors had assignments to simulate the operation of an ongoing system. One of these systems, which seemed to be a rather simple operation, was a self-service Laundromat with 10 washing machines and six dryers.



(a)



(b)

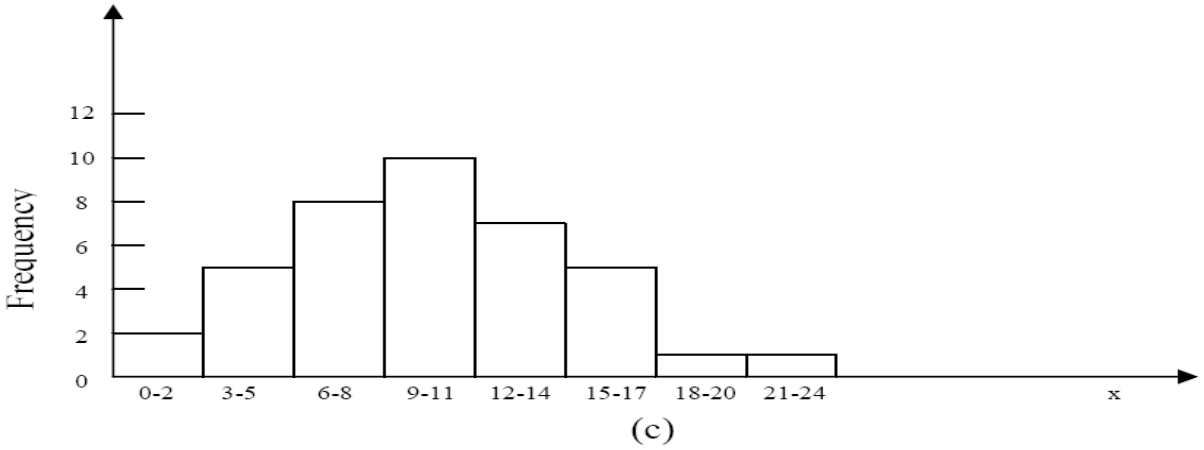


Fig 6.1 Ragged, coarse, and appropriate histogram : (a) original data- too ragged; (b) combining adjacent cells – too coarse; (c) combining adjacent cells – appropriate.

6.2 Identifying the Distribution with Data.

6.2.1 Histogram

A frequency distribution or histogram is useful in identifying the shape of a distribution.

A histogram is constructed as follows:

1. Divide the range of the data into intervals (intervals are usually of equal width; however, unequal widths however, unequal width may be used if the heights of the frequencies are adjusted).
2. Label the horizontal axis to conform to the intervals selected.
3. Determine the frequency of occurrences within each interval.
4. Label the vertical axis so that the total occurrences can be plotted for each interval.
5. Plot the frequencies on the vertical axis.

- If the intervals are too wide, the histogram will be coarse, or blocky, and its shape and other details will not show well. If the intervals are too narrow, the histogram will be ragged and will not smooth the data.
- The histogram for continuous data corresponds to the probability density function of a theoretical distribution.

Example 6.2 :

The number of vehicles arriving at the northwest corner of an intersection in a 5min period between 7 A.M. and 7:05 A.M. was monitored for five workdays over a 20-week period. Table shows the resulting data. The first entry in the table indicates that there were 12:5 min periods during which zero vehicles arrived, 10 periods during which one vehicles arrived, and so on,

Table 6:1 Number of Arrivals in a 5 Minute period

Arrivals Per period	Frequency	Arrivals Per Period	Frequency
0	12	6	7
1	10	7	5
2	19	8	5
3	17	9	3
4	10	10	3
5	8	11	1

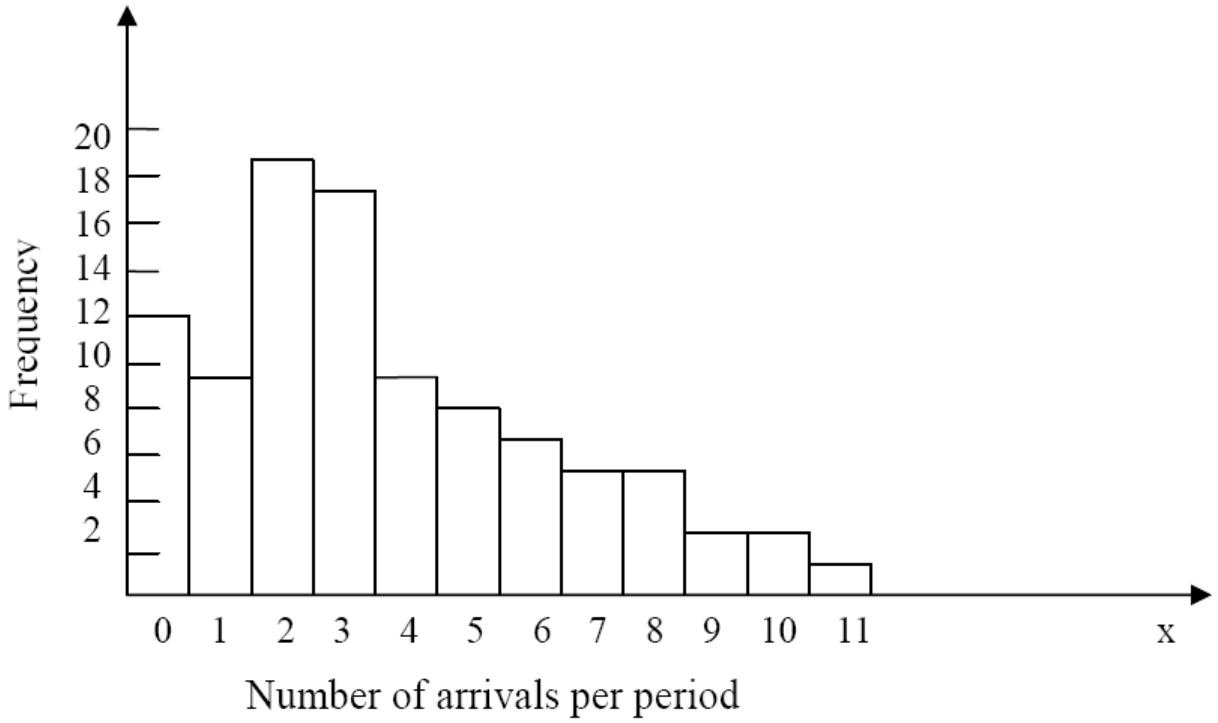


Fig 6.2 Histogram of number of arrivals per period.

6.2.2 Selecting the Family of Distributions

Additionally, the shapes of these distributions were displayed. The purpose of preparing histogram is to infer a known pdf or pmf. A family of distributions is selected on the basis of what might arise in the context being investigated along with the shape of the histogram.

Thus, if interarrival-time data have been collected, and the histogram has a shape similar to the pdf in Figure 5.9.the assumption of an exponential distribution would be warranted.

• Similarly, if measurements of weights of pallets of freight are being made, and the histogram appears symmetric about the mean with a shape like that shown in Fig 5.12, the assumption of a normal distribution would be warranted.

• The exponential, normal, and Poisson distributions are frequently encountered and are not difficult to analyze from a computational standpoint. Although more difficult to analyze, the gamma and Weibull distributions provide array of shapes, and should not be overlooked when modeling an underlying probabilistic process. Perhaps an exponential distribution was assumed,

but it was found not to fit the data. The next step would be to examine where the lack of fit occurred.

• If the lack of fit was in one of the tails of the distribution, perhaps a gamma or Weibull distribution would more adequately fit the data.

• Literally hundreds of probability distributions have been created, many with some specific physical process in mind. One aid to selecting distributions is to use the physical basis of the distributions as a guide. Here are some examples:

Binomial : Models the number of successes in n trials, when the trials are independent with common success probability, p; for example, the number of defective computer chips found in a lot of n chips.

Negative Binomial (includes the geometric distribution) : Models the number of trials required to achieve k successes; for example, the number of computer chips that we must inspect to find 4 defective chips.

Poisson : Models the number of independent events that occur in a fixed amount of time or space: for example, the number of customers that arrive to a store during 1 hour, or the number of defects found in 30 square meters of sheet metal.

Normal : Models the distribution of a process that can be thought of as the sum of a number of component processes; for example, the time to assemble a product which is the sum of the times required for each assembly operation. Notice that the normal distribution admits negative values, which may be-impossible for process times.

Lognormal : Models the distribution of a process that can be thought of as the product of (meaning to multiply together) a number of component processes; for example, the rate of return on an investment, when interest is compounded, is the product of the returns for a number of periods.

Exponential : Models the time between independent events, or a process time which is memoryless (knowing how much time has passed gives no information about how much additional time will pass before the process is complete); for example, the times between the arrivals of a large number of customers who act independently of each other.

Gamma : An extremely flexible distribution used to model nonnegative random variables. The gamma can be shifted away from 0 by adding a constant.

Beta : An extremely flexible distribution used to model bounded (fixed upper and lower limits) random variables. The beta can be shifted away from 0 by adding a constant and can have a larger range than [0,1] by multiplying by a constant.

Weibull : Models the time to failure for components; for example, the time to failure for a disk drive. The exponential is a special case of the Weibull. Discrete or Continuous Uniform Models complete uncertainty , since all outcomes are equally likely. This distribution is often overused when there are no data.

Triangular Models a process when only the rninimum, most-likely, and maximum values of the distribution are known; for example, the minimum, most- likely, and maximum time required to test a product.

Empirical Resamples from the actual data collected; often used when no theoretical distribution seems appropriate.

6.3 Parameter Estimation

After a family of distributions has been selected, the next step is to estimate the parameters of the distribution.

6.3.1 Preliminary Statistics: Sample Mean and Sample Variance

In a number of instances the sample mean, or the sample mean and sample variance, are used to estimate of the parameters of hypothesized distribution;
The **three** sets of equations are given for computing the sample mean and sample variance, -
1. *Equations (9.1) and (9.2) are used when discrete or continuous raw data are available*.
2. *Equations (9.3).and (9.4). are used when the data are discrete and have been grouped in frequency distribution*.
3. Equations (9.5) and (9.6) are used when the data are discrete or continuous and-have been placed in class intervals. Equations (9.5) and (9.6) are approximations and should be used only when the raw data are unavailable.
If the observations in a sample of size n are X1, X2,..., Xn, the sample mean (X) is defined by:

$$\overline{X} = \frac{\sum_{i=1}^{n} Xi}{n} \qquad 9.1$$

and the sample variance, s2 is defined by

$$S^2 = \frac{\sum_{i=1}^{n} Xi^2 - n\overline{X}^2}{n-1} \qquad 9.2$$

If the data are discrete and grouped in frequency distribution, Equation (9.1) and (.2) can be modified to provide for much greater computational efficiency, The sample mean can be computed by

$$\overline{X}^2 = \frac{\sum_{j=1}^{n} fjXj}{n} \qquad 9.3$$

And the sample variance by

$$X^2 = \frac{\sum_{j=1}^{k} fjXj^2 - n\overline{X}^2}{n-1} \qquad 9\ 4$$

Where k is the number of distinct values of X and fj is the observed frequency of the value Xj, of X.

6.3.2 Suggested Estimators

Numerical estimates of the distribution parameters are needed to reduce the family of distributions to a specific distribution and to test the resulting hypothesis.

- These estimators are the maximum-likelihood estimators based on the raw data. (If the data are in class intervals, these estimators must be modified.)
- The triangular distribution is usually employed when no data are available, with the parameters obtained from educated guesses for the minimum, most likely, and maximum possible value's; the uniform distribution may also be used in this way if only minimum and maximum values are available.
- Examples of the use of the estimators are given in the following paragraphs. The reader should keep in mind that a parameter is an unknown constant, but the estimator is a statistic or random variable because it depends on the sample values. To distinguish the two clearly, if, say, a parameter is denoted by a, the estimator will be denoted by _.

6.4 Goodness-of-Fit Tests

Chi-Square Test

One procedure for testing the hypothesis that a random sample of size *n* of the random variable *X* follows a specific distributional form is the chi-square goodness-of-fit test.

· This test formalizes the intuitive idea of comparing the histogram of the data to the shape of the candidate density or mass function, The test is valid for large sample sizes, for both discrete and continuous distribution assumptions, When parameters are estimated by maximum likelihood.

Where **0i**, is the observed frequency in the ith class interval and **Ei**, is the expected frequency in that class interval. The expected frequency for each class interval is computed as **Ei = npi,** where **pi** is the theoretical, hypothesized probability associated with the ith class interval.

□□It can be shown that approximately follows the chi-square distribution with k-s-1 degrees of freedom, where s represents the number of parameters of the hypothesized distribution estimated by sample statistics. The hypotheses are:

H0: the random variable, X, conforms to the distributional assumption with the parameter(s) given by the parameter estimate(s)

H1: the random variable X does not conform

□□If the distribution being tested is discrete; each value of the random variable should be a class interval, unless it is necessary to combine adjacent class intervals to meet the minimum expected cell-frequency requirement. For the discrete case, if combining adjacent cells is not required,

$$Pi = P(XI) = P(X = Xi)$$

Otherwise, **pi**, is determined by summing the probabilities of appropriate adjacent cells.

If the distribution being tested is continuous, the class intervals are given by [ai-1,ai), where **ai-1** and **ai**, are the endpoints of the *ith* class interval. For the continuous case with assumed pdf f(x), or assumed cdf *F(x),* pi, can be computed By

$$Pi = \int f(x)\,dx = F(ai) - F(ai - 1)$$

UNIT - 7 OUTPUT ANALYSIS FOR A SINGLE MODEL

Purpose

- Objective: Estimate system performance via simulation
- If q is the system performance, the precision of the estimator can be measured by:
  - The standard error of .
  - The width of a confidence interval (CI) for q.
- Purpose of statistical analysis:
  - To estimate the standard error or CI .
  - To figure out the number of observations required to achieve desired error/CI.
- Potential issues to overcome:
  - Autocorrelation, e.g. inventory cost for subsequent weeks lack statistical independence.
  - Initial conditions, e.g. inventory on hand and # of backorders at time 0 would most likely influence the performance of week 1.

7.1Type of Simulations

- Terminating verses non-terminating simulations
- Terminating simulation:
  - Runs for some duration of time T_E, where E is a specified event that stops the simulation.
  - Starts at time *0* under well-specified initial conditions.
  - Ends at the stopping time T_E.
  - Bank example: Opens at 8:30 am (time *0*) with no customers present and *8* of the *11* teller working (initial conditions), and closes at 4:30 pm (Time $T_E = 480$ minutes).
  - The simulation analyst chooses to consider it a terminating system because the object of interest is one day‘s operation.

7.2 Stochastic Nature of Output Data

- Model output consist of one or more random variables (r. v.) because the model is an input-output transformation and the input variables are r.v.‘s.
- M/G/1 queueing example:
  - Poisson arrival rate = *0.1* per minute;
    service time ~ *N(m = 9.5, s =1.75).*
  - System performance: long-run mean queue length, $L_Q(t)$.
  - Suppose we run a single simulation for a total of 5,000 minutes
    - Divide the time interval [*0, 5000*) into *5* equal subintervals of *1000* minutes.

Average number of customers in queue from time *(j-1)1000* to *j(1000)* is Y_j.

- M/G/1 queueing example (cont.):
  - Batched average queue length for 3 independent replications:

Batching Interval (minutes)	Batch, j	Replication 1, Y_{1j}	2, Y_{2j}	3, Y_{3j}
[0, 1000)	1	3.61	2.91	7.67
[1000, 2000)	2	3.21	9.00	19.53
[2000, 3000)	3	2.18	16.15	20.36
[3000, 4000)	4	6.92	24.53	8.11
[4000, 5000)	5	2.82	25.19	12.62
[0, 5000)		3.75	15.56	13.66

- Inherent variability in stochastic simulation both within a single replication and across different replications.
- The average across 3 replications, can be regarded as independent observations, but averages within a replication, $Y_{11}, \ldots, Y_{15}$, are not.

7.3 Measures of performance

- Consider the estimation of a performance parameter, q (or f), of a simulated system.
  - Discrete time data: $[Y_1, Y_2, \ldots, Y_n]$, with ordinary mean: q
  - Continuous-time data: $\{Y(t), 0 \le t \le T_E\}$ with time-weighted mean: f
- Point estimation for discrete time data.
  - The point estimator:

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

Is unbiased if its expected value is θ, that is if:

Is biased if: $E(\hat{\theta}) = \theta$

Point Estimator

- Point estimation for continuous-time data.
  - The point estimator:

$$\hat{\phi} = \frac{1}{T_E}\int_0^{T_E} Y(t)dt$$

- Is biased in general where: .
- An unbiased or low-bias estimator is desired.

- Usually, system performance measures can be put into the common framework of q or f:

e.g., the proportion of days on which sales are lost through an out-of-stock situation, let:

$$Y(t) = \begin{cases} 1, & \text{if out of stock on day } i \\ 0, & \text{otherwise} \end{cases}$$

- Performance measure that does not fit: quantile or percentile:
  - Estimating quantiles: the inverse of the problem of estimating a proportion or probability. $\Pr\{Y \le \theta\} = p$
  - Consider a histogram of the observed values Y:
    - Find such that $100p$% of the histogram is to the left of (smaller than) .

Confidence-Interval Estimation

- To understand confidence intervals fully, it is important to distinguish between measures of error, and measures of risk, e.g., confidence interval versus prediction interval.
- Suppose the model is the normal distribution with mean q, variance s^2 (both unknown).
  - Let Y_i be the average cycle time for parts produced on the i^{th} replication of the simulation (its mathematical expectation is q).
  - Average cycle time will vary from day to day, but over the long-run the average of the averages will be close to q.
  - Sample variance across R replications: $S^2 = \frac{1}{R-1}\sum_{i=1}^{R}(Y_i - Y_{\cdot\cdot})^2$

Confidence-Interval Estimation

- Confidence Interval (CI):
  - A measure of error.
  - Where $Y_{i\cdot}$ are normally distributed.

$$Y_{\cdot\cdot} \pm t_{\alpha/2,R-1}\frac{S}{\sqrt{R}}$$

- We cannot know for certain how far is from q but CI attempts to bound that error.
- A CI, such as 95%, tells us how much we can trust the interval to actually bound the error between and q .
- The more replications we make, the less error there is in (converging to *0* as *R* goes to infinity).

Confidence-Interval Estimation

- Prediction Interval (PI):
  - A measure of risk.
  - A good guess for the average cycle time on a particular day is our estimator but it is unlikely to be exactly right.
  - PI is designed to be wide enough to contain the *actual* average cycle time on any particular day with high probability.
  - Normal-theory prediction interval:

$$Y_{..} \pm t_{\alpha/2,R-1} S \sqrt{1+\frac{1}{R}}$$

  - The length of PI will not go to *0* as *R* increases because we can never simulate away risk.
  - PI's limit is: $\theta \pm z_{\alpha/2}\sigma$

7.4 Output Analysis for Terminating Simulations

- A terminating simulation: runs over a simulated time interval [*0, T_E*].

A common goal is to estimate:

$$\theta = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right), \quad \text{for discrete output}$$

$$\phi = E\left(\frac{1}{T_E}\int_0^{T_E} Y(t)dt\right), \quad \text{for continuous output } Y(t), 0 \le t \le T_E$$

- In general, independent replications are used, each run using a different random number stream and independently chosen initial conditions.

Statistical Background

- Important to distinguish within-replication data from across-replication data.
- For example, simulation of a manufacturing system

- Two performance measures of that system: cycle time for parts and work in process (WIP).
- Let Y_{ij} be the cycle time for the j^{th} part produced in the i^{th} replication.
- Across-replication data are formed by summarizing within-replication data .

- Across Replication:
  - For example: the daily cycle time averages (discrete time data)
    - The average:
    - The sample variance:
    - The confidence-interval half-width:
- Within replication:
  - For example: the WIP (a continuous time data)
    - The average:
    - The sample variance:

- Overall sample average, , and the interval replication sample averages, , are always unbiased estimators of the expected daily average cycle time or daily average WIP.
- Across-replication data are independent (different random numbers) and identically distributed (same model), but within-replication data do not have these properties.

UNIT 8

VERIFICATION AND VALIDATION OF SIMULATION MODELS OPTIMIZATION:

- One of the most important and difficult tasks facing a model developer is the verification and validation of the simulation model.
- It is the job of the model developer to work closely with the end users throughout the period.

Difference between Verification and Validation

Verification	Validation
• Verification is concerned with building the model right.	Validation is concerned with building the right model.
• It is utilized in comparison of the conceptual model to the computer representation that implements that conception.	• It is utilized to determine that a model is an accurate representation of the real system. It is usually achieved through the calibration of the model.
• The purpose of model verification is to assure that the conceptual model is reflected accurately in the operational model.	• Validation is the overall process of comparing the model and its behavior to the real system and its modeler.

8.1 Model Building, Verification, and Validation

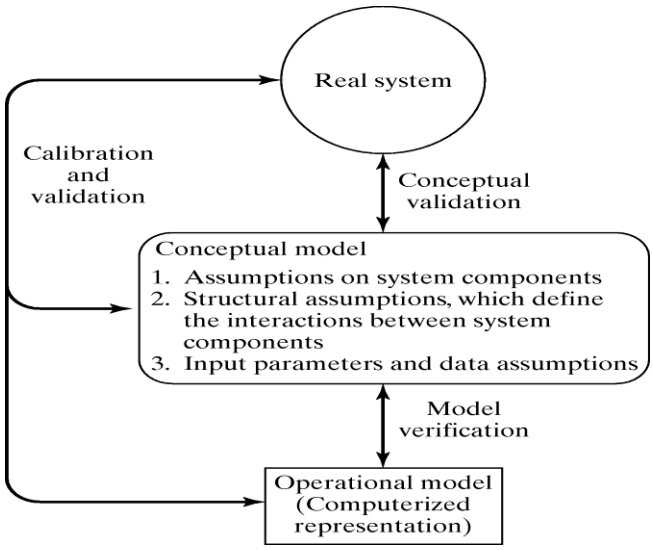


Fig: **Model Building, Verification, and Validation**

- The first step in model building consists of observing the real system and the interactions among its various components and collecting data on its behavior.
- Operators, technicians, repair and maintenance personnel, engineers, supervisors, and managers under certain aspects of the system which may be unfamiliar to others.
- As model development proceeds, new questions may arise, and the model developers will return, to this step of learning true system structure and behavior.
- The second step in model building is the construction of a conceptual model – a collection of assumptions on the components and the structure of the system, plus hypotheses on the values of model input parameters, illustrated by the following figure.

- The third step is the translation of the operational model into a computer recognizable form- the computerized model.

8.2**Verification of Simulation Models**

- The purpose of model verification is to assure that the conceptual model is reflected accurately in the computerized representation.
- The conceptual model quite often involves some degree of abstraction about system operations, or some amount of simplification of actual operations.

  Many suggestions can be given for use in the verification process:-

  1: Have the computerized representation checked by someone other than its developer.

  2: Make a flow diagram which includes each logically possible action a system can take when an event occurs, and follow the model logic for each a for each action for each event type.

  3: Closely examine the model output for reasonableness under a variety of settings of Input parameters.

  4. Have the computerized representation print the input parameters at the end of the Simulation to be sure that these parameter values have not been changed inadvertently.

  5. Make the computerized representation of self-documenting as possible.

  6. If the computerized representation is animated, verify that what is seen in the animation imitates the actual system.

  7. The interactive run controller (IRC) or debugger is an essential component of Successful simulation model building. Even the best of simulation analysts makes mistakes or commits logical errors when building a model. The IRC assists in finding and correcting those errors in the follow ways:

(a) The simulation can be monitored as it progresses.

(b) Attention can be focused on a particular line of logic or multiple lines of logic that constitute a procedure or a particular entity.

(c) Values of selected model components can be observed. When the simulation has paused, the current value or status of variables, attributes, queues, resources, counters, etc., can be observed.

(d) The simulation can be temporarily suspended, or paused, not only to view information but also to reassign values or redirect entities.

8. Graphical interfaces are recommended for accomplishing verification & validation .

8.3**Calibration and Validation of Models**

- Verification and validation although are conceptually distinct, usually are conducted simultaneously by the modeler.

- Validation is the overall process of comparing the model and its behavior to the real system and its behavior.

- Calibration is the iterative process of comparing the model to the real system, making adjustments to the model, comparing again and so on.

- The following figure 7.2 shows the relationship of the model calibration to the overall validation process.

- The comparison of the model to reality is carried out by variety of test.

- Tests are subjective and objective.
- Subjective test usually involve people, who are knowledgeable about one or more aspects of the system, making judgments about the model and its output.
- Objective tests always require data on the system's behavior plus the corresponding data produced by the model.

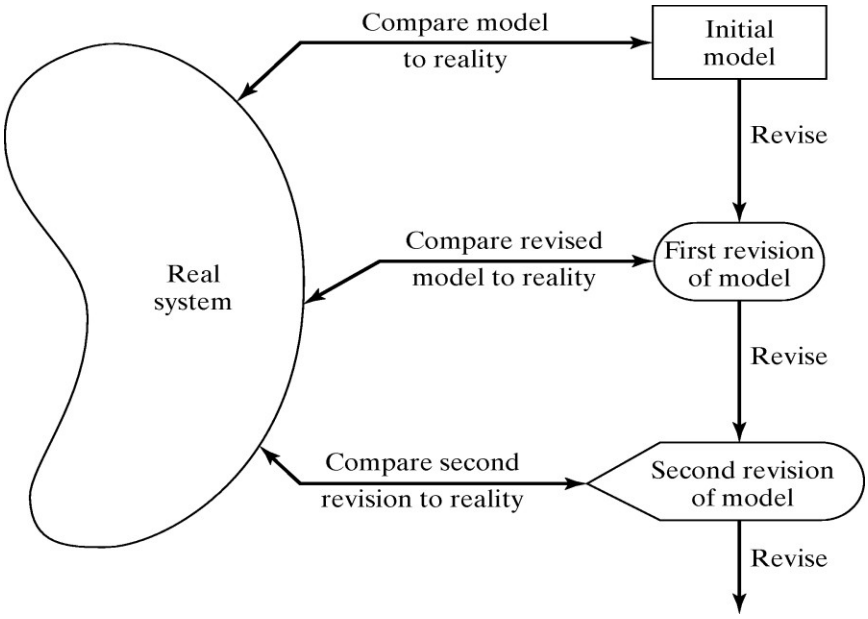


Fig. 8.2 Iterative process of calibration a model

- As an aid in the validation process, **Naylor and Finger** [1967] formulated a **three step approach** which has been widely followed:-

  1. Build a model that has high face validity.

  2. Validate model assumptions.

  3. Compare the model input-output transformations to corresponding input-output transformations for the real system.

8.3.1 FACE VALIDITY

- The first goal of the simulation modeler is to construct a model that appears reasonable on its face to model users and others who are knowledgeable about the real system being simulated.
- The users of a model should be involved in model construction from its conceptualization to its implementation to ensure that a high degree of realism is built into the model through reasonable assumptions regarding system structure, and reliable data.
- Another advantage of user involvement is the increase in the models perceived validity or credibility without which manager will not be willing to trust simulation results as the basis for decision making.
- Sensitivity analysis can also be used to check model's face validity.
- The model user is asked if the model behaves in the expected way when one or more input variables is changed.

- Based on experience and observations on the real system the model user and model builder would probably have some notion at least of the direction of change in model output when an input variable is increased or decreased.

- The model builder must attempt to choose the most critical input variables for testing if it is too expensive or time consuming to: vary all input variables.

8.3.2 Validation of Model Assumptions

- Model assumptions fall into two general classes: **structural assumptions and data assumptions.**

- **Structural assumptions** involve questions of how the system operates and usually involve simplification and abstractions of reality.

- For example, consider the customer queuing and service facility in a bank. Customers may form one line, or there may be an individual line for each teller. If there are many lines, customers may be served strictly on a first-come, first-served basis, or some customers may change lines if one is moving faster. The number of tellers may be fixed or variable. These structural assumptions should be verified by actual observation during appropriate time periods together with discussions with managers and tellers regarding bank policies and actual implementation of these policies.

- **Data assumptions** should be based on the collection of reliable data and correct statistical analysis of the data.

8.3.3 **Validating Input-Output Transformation**

- In this phase of validation process the model is viewed as input –output transformation.

- That is, the model accepts the values of input parameters and transforms these inputs into output measure of performance. It is this correspondence that is being validated.

- Instead of validating the model input-output transformation by predicting the future ,the modeler may use past historical data which has been served for validation purposes that is, if one set has been used to develop calibrate the model, its recommended that a separate data test be used as final validation test.

- Thus accurate **― prediction of the past‖** may replace prediction of the future for purpose of validating the future.

- A necessary condition for input-output transformation is that some version of the system under study exists so that the system data under at least one set of input condition can be collected to compare to model prediction.

- If the system is in planning stage and no system operating data can be collected, complete input-output validation is not possible.

- Validation increases modeler‘s confidence that the model of existing system is accurate.

- Changes in the computerized representation of the system, ranging from relatively minor to relatively major include :

  1: Minor changes of single numerical parameters such as speed of the machine, arrival rate of the customer etc.
  2: Minor changes of the form of a statistical distribution such as distribution of service time or a time to failure of a machine.
  3: Major changes in the logical structure of a subsystem such as change in queue discipline for waiting-line model, or a change in the scheduling rule for a job shop model.
  4: Major changes involving a different design for the new system such as computerized inventory control system replacing a non computerized system .